

Picard-Fuchs operators for octic arrangements I (*The case of orphans*)*

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We report on 25 families of projective Calabi-Yau threefolds that do not have a point of maximal unipotent monodromy in their moduli space. The construction is based on an analysis of certain pencils of octic arrangements that were found by C. MEYER [35]. There are seven cases where the Picard-Fuchs operator is of order two and 18 cases where it is of order four. The birational nature of the Picard-Fuchs operator can be used effectively to distinguish between families whose members have the same Hodge numbers.

1. Introduction

The phenomenon of mirror symmetry among Calabi-Yau threefolds has attracted a lot of attention and has led to major developments in mathematics and physics, see e.g. [26], [24]. Especially the marvelous discovery by P. CANDELAS, X. DE LA OSSA and coworkers [8] of the relation between enumeration of rational curves on a Calabi-Yau threefold and period integrals on another mirror manifold has been an inspiration to many researchers. Both the determination of instanton numbers in [8] and the construction of mirror pairs by V. BATYREV [5] as generalised by M. GROSS and B. SIEBERT [22], depend on *families* of Calabi-Yau manifolds that degenerate at the boundary of their moduli space at a point of *maximal unipotent monodromy* (MUM), [36]. In many natural families of Calabi-Yau threefolds, like hypersurfaces in toric varieties defined by reflexive polytopes, there do exist such MUM-points in their moduli space.

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However, it has been known for some time that there are quite simple examples of one-parameter families

$$f : \mathcal{Y} \longrightarrow S$$

of Calabi-Yau threefolds for which there are no such MUM points. First examples of this kind were described by J.-C. ROHDE [40] and later by A. GARBAGNATI and B. VAN GEEMEN [20]. However, these examples are somewhat atypical in the sense that the cohomological local system $\mathbb{H} = R^3 f_*(\mathbb{C})$ decomposes as a tensor product

$$\mathbb{H} = \mathbb{F} \otimes \mathbb{V},$$

where \mathbb{F} is a constant VHS with Hodge numbers $(1, 0, 1)$ and \mathbb{V} a variable $(1, 1)$ -VHS. As a consequence, although the cohomology space is four dimensional, the Picard-Fuchs operator is of order two, and ROHDE [41] asked the question if there exist examples where the Picard-Fuchs operator had order four.

In [47] we described an example of such a family, but the members of that family had the defect of not being projective. W. ZUDILIN [54] suggested to name *orphan* to describe such families, as they do not have a MUM. We decided to follow his suggestion, and in this paper we will describe a series of new orphans which have the virtue of being projective as well. The first such example was announced in [13] and has the full symplectic group $Sp_4(\mathbb{C})$ as differential Galois group.

Main result

There exist at least 10 birationally unrelated families of projective orphans with Picard-Fuchs operator of order four.

For a precise definition of the notion of *related families* we refer to the discussion in section 7.

We also want to point out two interesting phenomena that we discovered during the analysis of the examples.

- We found one new (conjectural) instance of a Calabi-Yau threefold with a Hilbert modular form of weight $(4, 2)$ and level $6\sqrt{2}$, much like the famous example of CONSANI and SCHOLTEN, [10]. (Hilbert modularity for that example was shown in [16].)
- We found an example of cohomology change in a family of Calabi-Yau threefolds at a point with monodromy of finite order. As a consequence, the central fibre of any semi-stable model is reducible, contrary to what happens in families of K3-surfaces, according to KULIKOV'S theorem.

2. Double octics

By a *double octic* we understand a double cover Y of \mathbb{P}^3 ramified over a surface of degree 8. It can be given by an equation of the form

$$u^2 = f_8(x, y, z, w)$$

and thus can be seen as a hypersurface in weighted projective space $\mathbb{P}(1^4, 4)$. For a general choice of the degree eight polynomial f_8 the variety Y is a smooth Calabi-Yau space with Hodge numbers $h^{11} = 1, h^{12} = 149$. When the octic f_8 is a product of eight linear factors, we speak of an *octic arrangement*. These form a nine dimensional sub-family and for these, the double cover Y has singularities at the intersections of the planes. In the generic such situation Y is singular along $8 \cdot 7/2 = 28$ lines, and by blowing up these lines (in any order) we obtain a smooth Calabi-Yau manifold \tilde{Y} with $h^{11} = 29, h^{12} = 9$.

By taking the eight planes in special positions, the double cover Y acquires further singularities. As explained in [35], if the arrangement does not have double planes, fourfold lines or sixfold points, there exist a diagram

$$\begin{array}{ccc} \widehat{Y} & \xrightarrow{\hat{\pi}} & Y \\ \downarrow 2:1 & & \downarrow 2:1 \\ \widehat{\mathbb{P}^3} & \xrightarrow{\pi} & \mathbb{P}^3 \end{array}$$

where π is a sequence of blow ups, $\hat{\pi}$ a crepant resolution, and the vertical maps are two-fold ramified covers.

In this way a myriad of different Calabi-Yau threefolds \widehat{Y} can be constructed. One of the nice things is that one can read off the Hodge number h^{12} as the dimension of the space of deformations of the arrangements that do not change the combinatorial type, [12]. In his doctoral thesis [35], C. MEYER found 450 combinatorially different octic arrangements, determined their Hodge numbers and started the study of their arithmetical properties.

Among these 450 arrangements there were 11 arrangements with $h^{12}(\widehat{Y}) = 0$, so lead to rigid Calabi-Yau threefolds and 63 one-parameter families of arrangements with $h^{12}(\widehat{Y}_t)$ leading to one-parameter families (all defined over \mathbb{Q}) of Calabi-Yau threefolds, parametrised by \mathbb{P}^1 : for general $t \in \mathbb{P}^1$, the crepant resolutions

$$\widehat{Y}_t \xrightarrow{\hat{\pi}} Y_t$$

of the double octics Y_t can be put together into a family $\overline{\mathcal{Y}}$ over $S := \mathbb{P}^1 \setminus \Sigma$, where $\Sigma \subset \mathbb{P}^1$ is a finite set of special values, where the combinatorial type of the arrangement changes. At these points, the configuration becomes rigid, or ceases to be of Calabi-Yau type: the arrangement contains a double plane, a fourfold line or a sixfold point. When we use the sequence of blow-ups to resolve the generic fibre and apply this to all members of the family, we arrive at a diagram of the form

$$\begin{array}{ccc} \widehat{\mathcal{Y}} & \hookrightarrow & \widehat{\overline{\mathcal{Y}}} \\ f \downarrow & & \downarrow \overline{f} \\ S & \hookrightarrow & \mathbb{P}^1 \end{array}$$

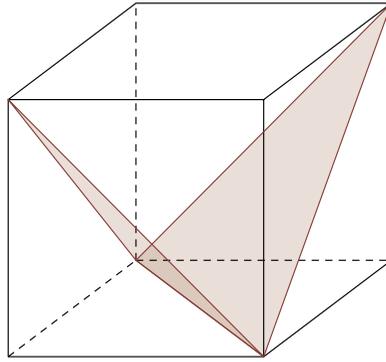
where f is a smooth map, with fibre \widehat{Y}_t , a smooth Calabi-Yau threefold with $h^{12} = 1$. In general, the fourfold $\widehat{\overline{\mathcal{Y}}}$ will have singularities sitting over the special points $s \in \Sigma$.

Recently, in [15] the analysis of MEYER was found to be complete and only three further examples with $h^{12} = 0$ exist over number fields and there is one further example of a family with $h^{12} = 1$, defined over $\mathbb{Q}(\sqrt{-3})$. Furthermore, in that paper various symmetries and birational maps between different arrangements were found.

We will take a closer look at these 63 MEYER-families. In order to facilitate comparison with literature, we will keep the numbering from [35]. In four cases (arrangements **33**, **155**, **275**, **276**) small adjustments in the parametrisation of the family were made.

Although arithmetical information on varieties in these families is readily available via the counting of points in finite fields, we found it extremely hard to understand details of the resolution and topology from the combinatorics of the arrangements. For example, the Jordan type of the local monodromy around the special points turned out to be very delicate. In particular, we failed to find a clear combinatorial way to recognise the appearance of a MUM-point.

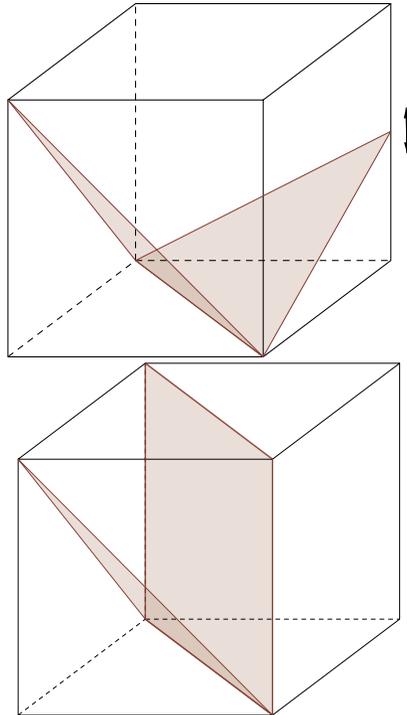
As an example, we look at configuration **69** of MEYER. It consists of six planes making up a cube, with two additional planes that pass through a face-diagonal and opposite vertices of the cube, as in the following picture.



Arrangement 69

The configuration is rigid and its resolution is a rigid Calabi-Yau with $h^{11} = 50, h^{12} = 0$.

By sliding the intersection point at the corner of one of the two planes, we arrive at configuration **70**, with $h^{11} = 49, h^{12} = 1$. Clearly, this pencil contains another rigid configuration, namely arrangement **3**.



Family of arrangements 70 and arrangement 3

But there are also two degenerations involving two double planes and it is not so clear what the corresponding fibres of the semi-stable reduction

look like, nor were we able to determine topologically the monodromy around these points. It turns out this family is one of the simplest orphans we know of and in this paper we will be dealing with the 25 cases without such a MUM-point and which thus do not make it onto the list of Calabi-Yau operators [3], [1], [9]. In the sequel [14] to this botanical paper, we will report on Picard-Fuchs operators for the remaining 38 MUM-cases.

3. (1, 1, 1, 1)-variations

In this section we recall some general facts on Variations of Hodge Structures that can be found for instance in the book by PETERS and STEENBRINK [38]. For the particular case of (1, 1, 1, 1)-variations we refer to the paper *Calabi-Yau operators* [48].

3.1. Generalities

Let us consider more generally a fibre square diagram

$$\begin{array}{ccc} \mathcal{Y} & \hookrightarrow & \bar{\mathcal{Y}} \\ f \downarrow & & \downarrow \bar{f} \\ S & \hookrightarrow & \mathbb{P}^1 \end{array}$$

where $f : \mathcal{Y} \rightarrow S := \mathbb{P}^1 \setminus \Sigma$ is a smooth proper map of Calabi-Yau threefolds. The datum of the local system

$$\mathbb{H}_{\mathbb{C}} := R^3 f_* \mathbb{C}_{\mathcal{Y}}$$

is equivalent, after choice of a base point $s \in S$, to that of its monodromy representation

$$\rho : \pi_1(\mathbb{P}^1 \setminus \Sigma, s) \rightarrow \text{Aut}(H^3(Y_s, \mathbb{C})).$$

There is an underlying lattice bundle $\mathbb{H}_{\mathbb{Z}}$, coming from the integral cohomology and a non-degenerate skew-symmetric intersection pairing, causing this representation to land in

$$\text{Aut}(H^3(Y_s, \mathbb{Z})/\text{torsion}) = \text{Sp}_m(\mathbb{Z}), \quad m = \dim H^3(Y_s).$$

Furthermore, \mathbb{H} carries the structure of a *polarised variation of Hodge structures (VHS)*, meaning basically that the fibres $\mathbb{H}_t = H^3(Y_t, \mathbb{C})$ of the

local system carry a pure (polarised) Hodge structure with Hodge numbers $(1, h^{12}, h^{12}, 1)$. It is a fundamental fact, proven by SCHMID [43], that one may complement this VHS defined on $\mathbb{P}^1 \setminus \Sigma$ by adding for each $s \in \Sigma$ a so called *Mixed Hodge Structure* (MHS) $(H_s, W_\bullet, F^\bullet)$. Here H_s is a \mathbb{Q} -vector space that can be identified with the sections of the \mathbb{Q} -local system $\mathbb{H}_{\mathbb{Q}}$ over an arbitrary small slit disc centered at s . The local monodromy transformation $T := T_s : H_s \rightarrow H_s$ at s can be written as

$$T = US$$

where U is unipotent and S is semi-simple. The *monodromy logarithm*

$$N = -\log U = (1 - U) + \frac{1}{2}(1 - U)^2 + \frac{1}{3}(1 - U)^3 + \dots$$

is nilpotent and determines a weight filtration W_\bullet on H_s which is characterised by the property that $N : W_k \rightarrow W_{k-2}$ and

$$N^k : Gr_{d+k}^W H_s \xrightarrow{\simeq} Gr_{d-k}^W H_s.$$

The Hodge filtration F^\bullet in $H_s \otimes \mathbb{C}$ arises as limit from the Hodge filtration on the spaces \mathbb{H}_t , when $t \mapsto s$, and it is a fundamental fact that for each $s \in \Sigma$ the F^\bullet defines a pure Hodge structure of weight k on the graded pieces $Gr_k^W H_s$.

In the geometrical case STEENBRINK [46] has constructed this mixed Hodge structure on H_s using a semi-stable model

$$\begin{array}{ccccc} D & \hookrightarrow & \mathcal{Z} & \longrightarrow & \overline{\mathcal{Y}} \\ \downarrow & & \downarrow & & \downarrow \overline{f} \\ \{s\} & \hookrightarrow & \Delta & \longrightarrow & \mathbb{P}^1 \end{array}$$

Here Δ is a small disc, $\Delta \rightarrow \mathbb{P}^1$ is a finite covering map, ramified over one of the $s \in \Sigma$, \mathcal{Z} is smooth and the fibre D over s is a (reduced) normal crossing divisor with components D_i inside \mathcal{Z} . The complex of relative logarithmic differential forms

$$\Omega_{\mathcal{Z}/\Delta}^\bullet(\log D)$$

can be used to describe the cohomology of the fibres and its extension to Δ . The complex comes with two filtrations F^\bullet, W_\bullet , which induces filtrations

on the hypercohomology groups

$$\mathbb{H}^d(\Omega_{\mathcal{Z}/\Delta}^\bullet(\log D) \otimes \mathcal{O}_D),$$

which then leads to the limiting mixed Hodge structure on H_s . We refer to [38] for a detailed account.

If the family $f : \mathcal{Y} \rightarrow S$ is defined over \mathbb{Q} , there is also a treasure of arithmetical information associated to the situation. We obtain for each rational point of $\mathbb{P}^1 \setminus \Sigma$ a Galois-representation on the l -adic cohomology $H_{\text{ét}}^3(Y_t \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_l)$, and these together make up an l -adic sheaf.

We will be mainly interested in the case where

$$h^{12} = 1,$$

so the local system \mathbb{H} on $\mathbb{P}^1 \setminus \Sigma$ is a so-called $(1, 1, 1, 1)$ -variations and its representations lands in $Sp_4(\mathbb{Z})$. In particular, for each of the 63 families of MEYER, we obtain a family of double octics

$$0 = u^2 - f_8(x, y, z, w; t).$$

By crepant resolution of the general fibre a family (dropping the earlier $\hat{\cdot}$) we obtain such families

$$f : \mathcal{Y} \rightarrow S := \mathbb{P}^1 \setminus \Sigma$$

and from it an associated $(1, 1, 1, 1)$ -variation over $\mathbb{P}^1 \setminus \Sigma$.

Strictly speaking, an arrangement defined by $f_8 = 0$ does not define a single family. If we multiply f_8 with a t -dependent function $\varphi(t)$, the family defined by

$$0 = u^2 - \varphi(t)f_8(x, y, z, w; t)$$

is said to be a *twist* of the family

$$0 = u^2 - f_8(x, y, z, w; t)$$

Although twisting can be crucially important, its effect is usually easy to analyse, and we will consider families differing by a twist as essentially the same.

3.2. Degenerations of (1,1,1,1)-variations

There are four possibilities for the mixed Hodge diamond of the limiting mixed Hodge structures appearing for (1, 1, 1, 1)-VHS. The k -th row (counted from the bottom) of the diamond gives the Hodge numbers of Gr_k^W ; the monodromy logarithm operator N acts in the vertical direction, shifting downwards by two rows. The definition of the weight filtration makes the diagram symmetric with respect to reflection in the central horizontal line, whereas complex conjugation is a symmetry of the Hodge-diamond along the central vertical axis. The numbers in each slope = 1 (so SW-NE-direction) row of the diagram have to add up to the corresponding Hodge number of the variation, so are all equal to 1 in our case. Moreover, by the definition of the weight filtration, the operator N has the property $N^k : Gr_{d+k}^W H_s \xrightarrow{\simeq} Gr_{d-k}^W H_s$, which implies that there cannot be 0 between two 1's in the same column. The cases that arise are:

F-point

$$\begin{array}{cccc}
 & & 0 & \\
 & & 0 & 0 \\
 & 0 & 0 & 0 \\
 1 & 1 & 1 & 1 \\
 & 0 & 0 & 0 \\
 & 0 & 0 & \\
 & & 0 &
 \end{array}$$

In this case $N = 0$, so this case occurs if and only if the monodromy is of *finite order*. The limiting mixed Hodge structure is in fact pure of weight three. Often an automorphism of finite order will split the Hodge structure Gr_3^W :

$$(1 \ 1 \ 1 \ 1) \mapsto (1 \ 0 \ 0 \ 1) + (0 \ 1 \ 1 \ 0)$$

On the arithmetic side, one expect that when this happens over \mathbb{Q} , the characteristic polynomial of Frobenius will factor as

$$(1 - pa_p T + p^3 T^2)(1 - c_p T + p^3 T^2)$$

where the a_p and c_p are Fourier coefficients of resp. a weight 2 and a weight 4 cusp form for some congruence subgroup $\Gamma_0(N)$ of $Sl_2(\mathbb{Z})$.

There are also cases where no splitting occurs, but the Euler-factors are determined by a Hilbert modular form of weight (4, 2) for some real quadratic extension of \mathbb{Q} .

C-point

$$\begin{array}{cccc}
& & & 0 \\
& & & 0 & 0 \\
& & 0 & 1 & 0 \\
1 & & 0 & 0 & 1 \\
& & 0 & 1 & 0 \\
& & 0 & 0 & \\
& & & & 0
\end{array}$$

In this case $N \neq 0$, $N^2 = 0$ and there is a single Jordan block. The pure part Gr_3^W is a rigid Hodge structure with Hodge numbers $(1, 0, 0, 1)$. Furthermore, Gr_4^W and Gr_2^W are one-dimensional and are identified via N . This type appears when a Calabi–Yau threefold acquires one or more ordinary double points, nowadays often called *conifold points*, which explains our name *C*-type point for it. However, one should be aware that *C*-point do occur not only where ordinary nodes appear, but also for many other kinds of singularities.

On the arithmetical side, one expects to get a 2-dimensional Galois representation Gr_3^W with characteristic polynomial of Frobenius of the form

$$1 - a_p T + p^3 T^2,$$

where the Frobenius traces are Fourier coefficient of a weight 4 cusp form for a congruence sub-group $\Gamma_0(N)$ of $Sl_2(\mathbb{Z})$, for some level N .

K-point

$$\begin{array}{cccc}
& & & 0 \\
& & & 0 & 0 \\
& & 1 & 0 & 1 \\
0 & & 0 & 0 & 0 \\
& & 1 & 0 & 1 \\
& & 0 & 0 & \\
& & & & 0
\end{array}$$

In this case we also have $N \neq 0$, $N^2 = 0$ but there are two Jordan blocks. In this case the pure part $Gr_3^W = 0$ and Gr_4^W , Gr_2^W are Hodge structures with Hodge numbers $(1, 0, 1)$, which are identified via N . The Hodge structure looks like that of the transcendental part of a K3-surface with maximal Picard number, which explains our name *K*-point for it.

On the arithmetical side, one expects to get a 2-dimensional Galois representation Gr_2^W with characteristic polynomial of Frobenius of the form

$$1 - a_p T + p^2 T^2,$$

where the Frobenius traces are Fourier coefficient of a weight 3 cusp form for a congruence sub-group $\Gamma_0(N)$ of $Sl_2(\mathbb{Z})$, for some level N and character. Such forms always have complex multiplication (CM). For a nice overview see [44] and [45].

MUM-point

$$\begin{array}{cccc} & & & 1 & \\ & & & 0 & 0 & \\ & & & 0 & 1 & 0 & \\ 0 & & & 0 & 0 & 0 & 0 & \\ & & & 0 & 1 & 0 & \\ & & & 0 & 0 & & \\ & & & & & & 1 & \end{array}$$

Here $N^3 \neq 0$ and there is a single Jordan block of maximal size. The Hodge structures Gr_{2k}^W ($k = 0, 1, 2, 3$) are one-dimensional and necessarily of Tate type. This happens for the quintic mirror at $t = 0$ and is one of the main defining properties of Calabi–Yau operators.

So at a MUM-point, the resulting mixed Hodge structure is an iterated extension of Tate–Hodge structures. DELIGNE [17] has shown that the instanton numbers n_1, n_2, n_3, \dots can be seen to encode precisely certain *extension data* attached to the variation of Hodge structures near the MUM-point.

4. Picard-Fuchs operators

4.1. Generalities

By a *choice of volume forms on the fibres of $\bar{f} : \bar{\mathcal{Y}} \rightarrow \mathbb{P}^1$* we mean a rational section ω of the relative dualising sheaf

$$\bar{f}_* \omega_{\bar{\mathcal{Y}}/\mathbb{P}^1}.$$

It restricts to a holomorphic 3-form $\omega(t)$ on each regular fibre Y_t outside the divisor of poles and zero's of ω . If $\gamma(t) \in H_3(Y_t, \mathbb{Z})$ is a family of horizontal cycles, defined in a contractible neighborhood U of $t \in S$, the function

$$\Phi_\gamma : U \rightarrow \mathbb{C}, t \mapsto \int_{\gamma(t)} \omega(t)$$

is called a *period integral* of $f : \mathcal{Y} \rightarrow \mathbb{P}^1$. It follows from the finiteness of the deRham cohomology group $H^3(Y_t, \mathbb{C})$ by *differentiation under the integral sign* that all period functions $\Phi_\gamma(t)$ satisfies the same linear ordinary differential equation, called the *Picard-Fuchs equation*. The corresponding differential operator is called *Picard-Fuchs operator* and we will write

$$\mathcal{P} = \mathcal{P}(\overline{\mathcal{Y}}, \omega) \in \mathbb{Q}\langle t, \frac{d}{dt} \rangle.$$

The order of the operator is clearly at most $\dim H^3(Y_t)$.

In the case of double octics given by an affine equation

$$u^2 - f_8(x, y, z, t)$$

we will always take the volume form

$$\omega := \frac{dx dy dz}{u}$$

as three-form and thus the period integrals we are dealing with are

$$\Phi_\gamma(t) := \int_{\gamma(t)} \frac{dx dy dz}{u} = \int_{\gamma(t)} \frac{dx dy dz}{\sqrt{f_8}}$$

over cycles $\gamma(t) \in H_3(Y_t, \mathbb{Z})$.

4.2. Determination of Picard-Fuchs operators

We have been using two fundamentally different methods to find Picard-Fuchs operators for concrete examples.

Conifold expansion method. If in a family of varieties we can locate a vanishing cycle, then the power series expansion of the period integral can always be computed algebraically, [13]. The operator is then found from the recursion of the coefficients.

Especially for the case of double octics, in many of the 63 families one can identify a *vanishing tetrahedron*: for a special value of the parameter one of the eight planes passes through a triple point of intersection, defined by three other planes. In appropriate coordinates we can write our affine equation as

$$u^2 = xyz(t - x - y - z)P(x, y, z, t)$$

where P is the product of the other five planes and depend on a the parameter t . We assume $P(0, 0, 0, 0) \neq 0$. One can now identify a nice cycle $\gamma(t)$ in the double octic, which consists of two parts $\gamma(t)_+$, ($u \geq 0$) and $\gamma(t)_-$, ($u \leq 0$) which project onto the real tetrahedron T_t bounded by the plane $x = 0, y = 0, z = 0, x + y + z = t$. $\gamma(t)_+$ and $\gamma(t)_-$ are glued together at the boundary of T_t , thus making up a three sphere in the double octic. For $t = 0$ the tetrahedron and thus the sphere $\Gamma(t)$ shrink to a point. We can write

$$\Phi_\gamma(t) = \int_{\gamma(t)} \omega = 2F(t)$$

where

$$\begin{aligned} F(t) &= \int_{T_t} \frac{dx dy dz}{\sqrt{(xyz(t-x-y-z)P(x,y,z,t))}} \\ &= t \int_T \frac{dx dy dz}{\sqrt{(xyz(1-x-y-z)P(tx,ty,tz,t))}}, \end{aligned}$$

where we used the substitution $(x, y, z) \mapsto (tx, ty, tz)$. By expanding the integrand in a series and perform termwise integration over the simplex $T := T_1$, the period expands in a series of the form

$$\Phi_\gamma(t) = \pi^2 t (A_0 + A_1 t + A_2 t^2 + \dots)$$

with the coefficients $A_i \in \mathbb{Q}$ if $P_t(x, y, z) \in \mathbb{Q}[x, y, z, t]$ and can be computed explicitly, see [13]. By computing sufficiently many terms in the expansion, one may find the Picard-Fuchs operator by looking for the recursion on the coefficients A_i .

Cohomology method. There are many variants for this method, but let us take for sake of simplicity the case of a smooth hypersurface $X \subset \mathbb{P}^n$. The middle dimensional (primitive) cohomology of X can be identified with $H^n(\mathbb{P}^n \setminus X)$ and its elements can be represented by residues of n -forms on \mathbb{P}^n with poles along X of the form

$$\frac{P\Omega}{F^k}$$

where F is the defining polynomial for X , $\Omega = \iota_E(dVol)$ the fundamental form for \mathbb{P}^n , and P is a polynomial such that the above expression is homogeneous of degree 0. If P runs over the appropriate graded pieces of the

Jacobi-ring of F , one obtains a basis

$$\omega_1, \omega_2, \dots, \omega_N$$

for $H^n(\mathbb{P}^n \setminus X)$. If F depends on an additional parameter t , one can differentiate these basis-elements with respect to t and express the result in the given basis. One obtains thus a differential system

$$\frac{d}{dt} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \dots \\ \omega_N \end{pmatrix} = A(t) \begin{pmatrix} \omega_1 \\ \omega_2 \\ \dots \\ \omega_N \end{pmatrix}$$

from which one can obtain Picard-Fuchs equations for each ω_i . This so-called *Griffiths-Dwork* method ([23], [18]) depends on the assumption that the hypersurface is smooth, so that the partial derivatives $\partial_i F$ form a regular sequence in the polynomial ring. If X has singularities, one no longer obtains a basis for the cohomology. Rather one has to determine the Koszul-homology between the partial derivatives and enter into a spectral sequence and things become more complicated. In [13] we wrote:

Due to the singularities of f_8 , a Griffiths-Dwork approach is cumbersome, if not impossible.

It was P. LAIREZ who proved us very wrong in this respect. His computer program, described in [34], does not aim at finding a complete cohomology space, rather it looks for the smallest space stable under differentiation that contains the given rational differential form. It does so by going through the spectral sequence given by the pole order filtration, where at each step Gröbner basis calculations are done to increase the set of basis forms.

For our computations we initially used the method of conifold expansion, but we discovered soon that

Due to the singularities of f_8 the conifold expansion approach is cumbersome, if not impossible.

The conifold expansion method has some serious drawbacks. In our implementation computations became too time consuming for operators of high degree and it could be applied only to families of double octics admitting a *clean vanishing tetrahedron*, as explained in [13], which happens only for about half of the cases. All operators included in the current paper were

computed with the package of PIERRE LAIREZ. As conifold expansion requires to bring the equation into a prepared form, the computed operators differ by a coordinate transformation from operators in [14], cf. operator 70.

4.3. Reading the Riemann-symbol

The amount of information that is contained in the Picard-Fuchs operator \mathcal{P} can not be underestimated. The local system $\mathbb{H}_{\mathbb{C}}$ is isomorphic to local system of solutions $Sol(\mathcal{P})$. Already the local monodromies T_s around the special parameter values $s \in \Sigma$ are hard to obtain from topology or a semi-stable reduction. But this information can easily be read off from the operator, by studying the local solutions in series of the form

$$t^\alpha \sum_{k=0}^N \sum_{n=0}^{\infty} A_{n,k} t^n \log(t)^k.$$

There is a delicate interaction between \mathcal{P} and the Frobenius-operator (see [19]), so that arithmetical properties of the varieties are tightly linked to \mathcal{P} . It appears that the Picard-Fuchs operator just abstracts away sufficiently many details of the geometry and retains just the right amount of motivic information.

We recall that the *Riemann-symbol* of a differential operator $\mathcal{P} \in \mathbb{C}\langle t, \frac{d}{dt} \rangle$ is a table recording for each singular point of the differential operator the corresponding *exponents*, i.e. solutions to the *indicial equation* [27]. (In order to have a non-zero series solution of the type described above, one needs that α is an exponent at 0.)

We found it convenient to express the operators in terms of the logarithmic differentiation

$$\Theta := t \frac{d}{dt}$$

and write the operators in Θ -form

$$\mathcal{P} := P_0(\Theta) + tP_1(\Theta) + t^2P_2(\Theta) + \dots t^rP_r(\Theta), \quad P_r \neq 0$$

where the P_i are polynomials, in our case of degree four. The exponents at 0 are then just the roots of P_0 , those of ∞ the roots of P_r , with a minus sign. To determine the exponents at other points, one just translate this point to the origin, and re-express the operator in Θ -form.

The exponents capture the semi-simple part of the monodromy at the corresponding singular point. The logarithmic terms appearing in the solutions encode the Jordan structure of the unipotent part. In general logarithmic terms may appear between solutions with integer difference in exponents, but in the geometrical context, as a rule, a logarithm appears *always* precisely when two exponents become *equal*.

A **C-point** can be expected when the exponent are of the form

$$\alpha - \epsilon \quad \alpha \quad \alpha \quad \alpha + \epsilon$$

The archetypical case is $0 \quad 1 \quad 1 \quad 2$, indicating the presence of local solutions of the form

$$\begin{aligned} \phi_0 &= 1 + a_1 t + \dots, \phi_1(t) = t + b_1 t + \dots, \phi_2(t) = \log(t)\phi_1(t) + c_1 t + \dots, \\ \phi_3 &= t^2 + d_1 t^3 + \dots \end{aligned}$$

as appear in a pure conifold smoothing.

A **K-point** can be expected to appear when the exponents are of the form

$$\alpha \quad \alpha \quad \beta \quad \beta, \quad (\alpha < \beta).$$

The archetypical case is $0 \quad 0 \quad 1 \quad 1$, indicating the presence of local solutions of the form

$$\begin{aligned} \phi_0 &= 1 + a_1 t + \dots, \phi_1(t) = \log(t)\phi_0(t) + b_0 + b_1 t + \dots, \\ \phi_2(t) &= t + c_1 t + \dots, \phi_3 = \log(t)\phi_1(t) + d_1 t^1 + \dots \end{aligned}$$

A **MUM-point** can be expected to appear, when all exponents are equal

$$\alpha \quad \alpha \quad \alpha \quad \alpha$$

The archetypical case is $0 \quad 0 \quad 0 \quad 0$, with the famous Frobenius basis of solutions of the form

$$\begin{aligned} \phi_0 &= 1 + at + \dots, \phi_1(t) = \log(t)\phi_0(t) + \dots, \phi_2(t) = \log(t)^2\phi_0(t) + \dots, \\ \phi_3(t) &= \log^3(t)\phi_0(t) + \dots \end{aligned}$$

An **F-point** can be expected in all other cases. If all exponents are integral (and no logarithms appear) we have trivial monodromy and we speak of an *apparent singularity*, if the exponents are *not* $0 \quad 1 \quad 2 \quad 3$,

which would be the exponents at a regular point. The archetypical apparent singularity is signaled by the exponents

$$0 \ 1 \ 3 \ 4.$$

In general, we will call any F -point *with non-equally spaced exponents* an A -point.

When we write the operator \mathcal{P} in the form

$$\frac{d^4}{dt^4} + a_1(t)\frac{d^3}{dt^3} + a_2(t)\frac{d^2}{dt^2} + a_3(t)\frac{d}{dt} + a_4(t)$$

where $a_i(t) \in \mathbb{C}(t)$, then the function

$$Y(t) := e^{-\frac{1}{2} \int a_1(t) dt}$$

is called the *Yukawa coupling* and its (simple) zero's typically are apparent singularities with exponents 0 1 3 4.

5. Birational nature of the Picard-Fuchs operator

5.1. Basic transformation theory

In what follows, we will not distinguish between an operator \mathcal{P} and the operator $\mathcal{P}' = \varphi(t)\mathcal{P}$ obtained from multiplying \mathcal{P} by a rational function $\varphi(t)$, as they determine the same local system of solutions on $\mathbb{P}^1 \setminus \Sigma$. (Of course, in the finer theory of \mathcal{D} -modules one has to distinguish very well between \mathcal{P} and \mathcal{P}').

Often one has to make simple transformations on differential operators.

1) The simplest are those induced by Möbius transformations, coming from fractional linear transformations

$$t \mapsto \frac{at + b}{ct + d}$$

of the coordinate in \mathbb{P}^1 . Of course, this just changes the position of the singular points, the corresponding exponents are preserved. We call two operators related in this way *similar operators*.

2) If $\omega(t)$ and $\omega'(t)$ are two different choices of volume form on the fibres, then

$$\omega'(t) = \varphi(t)\omega(t)$$

where $\varphi(t)$ is a rational function. The corresponding Picard-Fuchs operators $\mathcal{P}(\mathcal{Y}, \omega)$ and $\mathcal{P}(\mathcal{Y}, \omega')$ will be related in a certain way.

More generally, if $y(t)$ satisfies $\mathcal{P}y(t) = 0$, and $\phi(t)$ is a rational function, then the function $Y(t) := \phi(t)y(t)$ will satisfy another differential equation $\mathcal{Q}Y(t) = 0$ that is rather easy to determine. We will say that \mathcal{Q} is *strictly equivalent* to \mathcal{P} . Its effect on the Riemann symbol will be a shift of all exponents by an amount given by the order of $\phi(t)$ at the point in question. For example, the effect of multiplication by t shifts the exponents at 0 one up, those at ∞ one down.

3) As already mentioned above, if $\phi(t)$ is a rational function of t , then the families of double octics

$$u^2 = f_8, \quad \text{and} \quad u^2 = \phi(t)f_8$$

are said to differ by a *twist*. Replacing $\phi(t)$ by $\phi(t)\varphi(t)^2$ does not change the fibration birationally, as can be seen by replacing u by $\varphi(t)u$. The volume form

$$\omega := \frac{dx dy dz}{\sqrt{f_8}}$$

for $u^2 = f_8$ and

$$\omega' := \frac{dx dy dz}{\sqrt{\phi(t)f_8}}$$

for $u^2 = \phi(t)f_8$ differ by the square root of a rational function

$$\omega = \sqrt{\phi(t)}\omega'$$

More generally, if $y(t)$ satisfies a differential equation $\mathcal{P}y(t) = 0$, then $Y(t) := \phi(t)y(t)$, where $\phi(t)$ is an algebraic function, will satisfy another differential equation \mathcal{Q} that is easy to determine, knowing only \mathcal{P} and $\phi(t)$. We will then call \mathcal{P} and \mathcal{Q} *equivalent*.

Its effect on the Riemann symbol is also rather easy to understand. If $\phi(t)$ has near a the character of $(t - a)^\epsilon$, then the exponents at a get all shifted by the amount ϵ :

$$\alpha, \beta, \gamma, \delta \mapsto \alpha + \epsilon, \beta + \epsilon, \gamma + \epsilon, \delta + \epsilon$$

4) If we replace t by $t = \psi(s)$ for some function $\psi(s)$ we can rewrite the operator \mathcal{P} in terms of the variable s and obtain an operator $\psi^*\mathcal{P}$ in $s, \frac{d}{ds}$ that we call the *pull-back* of \mathcal{P} along the map ψ . Very common are pull-backs by the map $t = s^n$, which geometrically is an n -fold covering map of \mathbb{P}^1 , with total ramification at 0 and ∞ . This operation leads to a division of the exponents at 0 and ∞ :

$$\alpha, \beta, \gamma, \delta \mapsto \alpha/n, \beta/n, \gamma/n, \delta/n$$

The most general transformations one has to allow are those given by algebraic coordinate transformations, which are multi-valued maps from \mathbb{P}^1 to itself. To formulate this properly we need to extend our framework to include families and operators over an arbitrary algebraic curve.

5.2. Related operators, related families

Let S be an algebraic curve, K_S its function field, δ a derivation of K_S (a rational vector field), we denote by $K_S\langle\delta\rangle$ the differential algebra generated by K_S and δ . An element of $K_S\langle\delta\rangle$ can be written as

$$\mathcal{P} = a_n\delta^n + \cdots + a_1\delta + a_0$$

and is called a *differential operator of order n* on S .

Definition. Let $\mathcal{X} \rightarrow S$ be a family of Calabi-Yau manifolds, ω , a rational section of the relative dualizing sheaf of $\omega_{\mathcal{X}/S}$ and δ a derivation of K_S . A *Picard-Fuchs operator* of $(\mathcal{X}, \omega, \delta)$ is an operator $\mathcal{P}(\mathcal{X}, \omega, \delta) \in K_S\langle\delta\rangle$ of lowest order annihilating all period integrals $\int_{\gamma} \omega$. Such an operator is unique up to left multiplication by an element of K_S .

Definition. Two differential operators $\mathcal{P}_1, \mathcal{P}_2 \in K_S\langle\delta\rangle$ are called *strictly equivalent* if there exist $v_1, v_2 \in K_S \setminus \{0\}$ such that for any $u \in K_S$ we have

$$\mathcal{P}_1(v_1u) = v_2\mathcal{P}_2(u).$$

If $\Phi : S_1 \rightarrow S_2$ is a dominating morphism of irreducible algebraic curves, then $\Phi^*K_{S_2} \subset K_{S_1}$ is an algebraic field extension and consequently for any derivation δ of K_{S_2} there exists a unique derivation $\Phi^*\delta$ of K_{S_1} satisfying $(\Phi^*\delta)(\Phi^*f) = \Phi^*(\delta(f))$ for any $f \in K_{S_2}$.

Given a family of Calabi-Yau manifolds \mathcal{X}/S then any two rational sections ω and ω' of relative canonical bundle differ by a multiplication by an element of K_S , as a consequence the Picard-Fuchs operators

$$\mathcal{P}(\mathcal{X}, \omega, \delta) \quad \text{and} \quad \mathcal{P}(\mathcal{X}, \omega', \delta)$$

are strictly equivalent.

Definition. The pullback of an operator

$$\mathcal{P} = a_n \delta^n + \cdots + a_1 \delta + a_0 \in K_{S_2} \langle \delta \rangle$$

on an algebraic curve by a dominating morphism $\Phi : S_1 \rightarrow S_2$ is the operator

$$\Phi^* \mathcal{P} = \Phi^* a_n \Phi^* \delta^n + \cdots + \Phi^* a_1 \Phi^* \delta + \Phi^* a_0 \in K_{S_1} \langle \Phi^* \delta \rangle.$$

Pullback $\Phi^* \mathcal{P}$ is the unique operator in $K_{S_1} \langle \Phi^* \delta \rangle$ such that $\Phi^* \mathcal{P}(\Phi^* f) = \Phi^*(\mathcal{P}(f))$.

If $\mathcal{X} \rightarrow S_2$ is a family Calabi-Yau manifolds, $\Phi : S_1 \rightarrow S_2$ is a dominating morphism of algebraic curves then

$$\Phi^* \mathcal{P}(\mathcal{X}, \omega, \delta) = \mathcal{P}(\Phi^* \mathcal{X}, \Phi^* \omega, \Phi^* \delta).$$

Definition. Two differential operators $\mathcal{P}_1 \in K_{S_1} \langle \delta_1 \rangle$, $\mathcal{P}_2 \in K_{S_2} \langle \delta_2 \rangle$, are called *related* if there exist a curve S and dominating morphisms

$$\begin{array}{ccc} S & \xrightarrow{p_1} & S_1 \\ p_2 \downarrow & & \\ & & S_2 \end{array}$$

and $v_1, v_2 \in K_S$ such that for any $u \in K_S$

$$p_1^* \mathcal{P}_1(v_1 u) = v_2 p_2^* \mathcal{P}_2(u).$$

Definition. Two projective families $f_1 : \mathcal{X}_1 \rightarrow S_1$ and $f_2 : \mathcal{X}_2 \rightarrow S_2$ are called *related*, if there exist a curve S and dominating maps $g_1 : S \rightarrow S_1$, $g_2 : S \rightarrow S_2$ and an isomorphism

$$\varphi : g_1^* \mathcal{X}_1 \rightarrow g_2^* \mathcal{X}_2.$$

Clearly, if the families $\mathcal{X} \rightarrow S_1$ and $\mathcal{X} \rightarrow S_2$ are both obtained as pull-back from a single family over T , then the families are related.

Proposition 1. *Picard-Fuchs operators of related families of Calabi-Yau manifolds are related operators.*

5.3. Birational maps and strict equivalence

We would like to formulate a statement expressing the idea that the Picard-Fuchs operator is in some sense a birational invariant of a family. We formulate three theorems to this effect.

Proposition 2. *Consider $\mathcal{X}_1 \rightarrow S$ and $\mathcal{X}_2 \rightarrow S$ two proper smooth families of Calabi-Yau manifolds over an algebraic curve S . Let ω_1 and ω_2 be holomorphic volume forms on \mathcal{X}_1 and \mathcal{X}_2 . If there is a birational map*

$$\phi : \mathcal{X}_1 \dashrightarrow \mathcal{X}_2,$$

then the Picard-Fuchs operators $\mathcal{P}(\mathcal{X}_1, \omega_1, \delta)$ and $\mathcal{P}(\mathcal{X}_2, \omega_2, \delta)$ are strictly equivalent.

Proof. Let X_1 and X_2 be fibres over the same general point. Recall that V. Batyrev [6] proved that if

$$\phi : X_1 \dashrightarrow X_2$$

is a birational map between smooth Calabi-Yau manifolds, then it induces isomorphisms

$$\phi^* : H^{p,q}(X_2) \rightarrow H^{p,q}(X_1)$$

of Hodge groups. As a consequence, ϕ induces isomorphisms

$$\phi_* : H_n(X_1, \mathbb{Q}) \rightarrow H_n(X_2, \mathbb{Q}), \quad \phi^* : H^n(X_2, \mathbb{Q}) \rightarrow H^n(X_1, \mathbb{Q})$$

If $\omega_1 \in H^{n,0}(X_1) = H^0(X_1, \Omega_{X_1}^n)$ and $\omega_2 \in H^{n,0}(X_2) = H^0(X_2, \Omega_{X_2}^n)$ are holomorphic volume forms on X_1 and X_2 , then

$$\phi^*(\omega_2) = \lambda \omega_1$$

for some $\lambda \in \mathbb{C}^*$. Furthermore, one has

$$\int_{\gamma} \phi^*(\omega_2) = \int_{\phi_*(\gamma)} \omega_2$$

In the relative situation we consider a birational map that induces fibre-wise birational isomorphism

$$\phi : \mathcal{X}_1 \dashrightarrow \mathcal{X}_2$$

If $\omega_1(t)$ and $\omega_2(t)$ are volume forms on \mathcal{X}_1 resp. \mathcal{X}_2 , then

$$\phi^*(\omega_2(t)) = \lambda(t)\omega_1(t)$$

for some rational function $\lambda(t)$. So we have

$$\lambda(t) \int_{\gamma(t)} \omega_1(t) = \int_{\gamma(t)} \phi^*(\omega_2(t)) = \int_{\phi_*(\gamma(t))} \omega_2(t)$$

which shows that the period integrals for \mathcal{X}_1 and \mathcal{X}_2 differ by multiplication by a rational function. So the Picard-Fuchs operators

$$\mathcal{P}(\mathcal{X}_1, \omega_1, \delta) \quad \text{and} \quad \mathcal{P}(\mathcal{X}_2, \omega_2, \delta)$$

are strictly equivalent. □

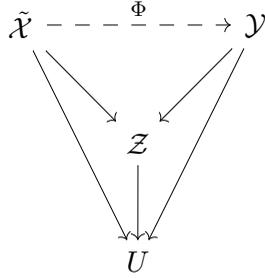
Proposition 3. *Let $\varphi : X \dashrightarrow Y$ be a birational map between two Calabi-Yau threefolds and $f : \mathcal{X} \rightarrow S$ be a family of Calabi-Yau threefolds such that X is isomorphic to the fibre \mathcal{X}_{s_0} of \mathcal{X} at some point $s_0 \in S$, then there exist étale neighbourhoods (U, u_0) of $s_0 \in S$, a family $g : \mathcal{Y} \rightarrow U$ with Y isomorphic to the fibre \mathcal{Y}_{u_0} and a fibre-wise birational map $\Phi : \mathcal{X}_U \dashrightarrow \mathcal{Y}$ that induces the birational map $\varphi : X \dashrightarrow Y$ at u_0 .*

Proof. As any birational map between smooth Calabi-Yau threefolds is a composition of flops ([30, Thm. 4.9]) we can consider only the special case when ϕ is a single flop, in this situation we have a diagram

$$\begin{array}{ccc} X & \dashrightarrow^{\varphi} & Y \\ & \searrow & \swarrow \\ & Z & \end{array}$$

where $X \rightarrow Z$ and $Y \rightarrow Z$ are contraction maps. By Proposition 12.2.8 from [31], there exists a finite étale map $g : V \rightarrow S$ such that the sheaf of flat families of 1-cycles with nonnegative coefficients modulo fiberwise numerical equivalence $\mathcal{GN}_1(g^*X/V)$, defined in 12.2.7 in [31], is a local system with trivial monodromy. Let $\tilde{\mathcal{X}} := g^*\mathcal{X}$ be the pullback of \mathcal{X} to V . Then we get a contraction morphism $\tilde{\mathcal{X}} \rightarrow \mathcal{Z}$ to a family $\mathcal{Z} \rightarrow V$. Let U be the set of points $u \in V$ where the exceptional locus of the contraction $\tilde{\mathcal{X}}_u \rightarrow \mathcal{Z}_u$ has dimension 1, U is open and there exists $u_0 \in U$ with $g(u_0) = s_0$.

By Theorem 11.10 from [31] there exists a flop Φ over U



that extends the flop φ , consequently \mathcal{Y} is a family of Calabi-Yau threefolds, Y is isomorphic to the fibre \mathcal{Y}_{u_0} of \mathcal{Y} and Φ is a fiberwise birational map. \square

5.4. Moduli spaces

We consider a smooth Calabi-Yau threefold X with $h^{1,2} = 1$. By the famous theorem of BOGOMOLOV, TIAN [50] and TODOROV [51], the local deformation theory of any Calabi-Yau manifold is unobstructed, and so by the classical deformation theory of KODAIRA, SPENCER [29] and KURANISHI [32] X posses a versal deformation over a smooth 1-dimensional disc Δ :

$$\begin{array}{ccc} X & \hookrightarrow & \mathcal{X} \\ \downarrow & & \downarrow f \\ \{0\} & \hookrightarrow & \Delta \end{array}$$

Even if X is projective there will not exist a well-defined moduli space for X , but rather one has a moduli stack. However, when we choose an ample line bundle L on X , we can form families over corresponding quasi projective moduli spaces, see [52]. As a result, we may produce many a priori different projective families

$$f : \mathcal{X}_L \longrightarrow S_L$$

which all have X as fibre. Now versality of the Kuranishi family $\mathcal{X} \longrightarrow \Delta$ implies that if X appears as fibre $f^{-1}(s)$ of any projective family $f : \mathcal{X}' \longrightarrow S$ over a curve S , and having a non-zero Kodaira-Spencer map at s , then this family is locally analytic isomorphic to the above model family $\mathcal{X} \longrightarrow \Delta$.

Theorem 4. *Let $f_1 : \mathcal{X}_1 \rightarrow S_1$ and $f_2 : \mathcal{X}_2 \rightarrow S_2$ be two projective families and*

$$\hat{\phi} : \hat{X}_1 \rightarrow \hat{X}_2$$

an isomorphism between the formal neighbourhoods \hat{X}_i of a fibre $X_i = f_i^{-1}(s_i) \subset \mathcal{X}_i$, then there exist isomorphisms ϕ, ψ

$$\begin{array}{ccc} U_1 & \xrightarrow{\phi} & U_2 \\ f_1 \downarrow & & \downarrow f_2 \\ V_1 & \xrightarrow{\psi} & V_2 \end{array}$$

of étale neighbourhoods U_i of $X_i \subset \mathcal{X}_i$, $s_i \in S_i$ ($i = 1, 2$) and thus the families $\mathcal{X}_1 \rightarrow S_1$ and $\mathcal{X}_2 \rightarrow S_2$ are related.

Proof. If the families \mathcal{X}_1 and \mathcal{X}_2 are versal at points s_1 and s_2 , then this is a particular case of a very general Theorem (1.7) (Uniqueness) proven by ARTIN in [4]. It states that if F is a functor of locally of finite presentation and $\bar{\xi} \in F(\bar{A})$ an effective versal family, then the triple (X, x, ξ) is unique up to local isomorphism for the étale topology, meaning that if (X', x', ξ') is another algebraisation, then there is a third one, dominating both. In our application we use only this special case.

In the general situation a proof can be given by a straightforward application of the *nested approximation theorem* as stated in [42, Thm. 5.8] (see also [33]). In the above situation it can be applied as follows: assume we have two algebraisations $\mathcal{X}_1 \rightarrow S_1$, $\mathcal{X}_2 \rightarrow S_2$, given by equations of the form $F(x, t) = 0$, $G(x', t') = 0$. We are looking for algebraic maps

$$\phi(x, t), \quad \psi(t)$$

that map \mathcal{X}_1 to \mathcal{X}_2 and S_1 to S_2 . Hence we look for solutions to the system of equations

$$F(x, t) = 0, \quad G(\phi(x, t), \psi(t)) = 0$$

Now as $\hat{\phi}$ is a formal equivalence between \mathcal{X}_1 and \mathcal{X}_2 it gives a formal solutions ϕ, ψ to the above equations and hence by the nested approximation theorem we obtain a solution in the ring of algebraic power series. \square

Lemma 5. *Let $f_1 : \mathcal{X}_1 \rightarrow S_1$ and $f_2 : \mathcal{X}_2 \rightarrow S_2$ be two families of Calabi-Yau threefolds over smooth quasi-projective curves versal at a generic point and*

$$\varphi : X_1 \rightarrow X_2$$

an isomorphism between fibres $X_i = f_i^{-1}(s_i) \subset \mathcal{X}_i$, then the families $\mathcal{X}_1 \rightarrow S_1$ and $\mathcal{X}_2 \rightarrow S_2$ are related.

Proof. Let $f : \mathcal{X} \rightarrow \Delta$ be a versal family of X , in appropriate local analytic coordinates at s_1 and s_2 the families \mathcal{X}_1 and \mathcal{X}_2 are locally pullbacks of \mathcal{X} by maps $t \mapsto t^a$ and $t \mapsto t^b$. Let $g_1 : \tilde{S}_1 \rightarrow \tilde{S}_1$ be a covering of degree b totally ramified at s_1 , $g_2 : \tilde{S}_2 \rightarrow \tilde{S}_2$ be a covering of degree a totally ramified at s_2 . Then the pullback families $\tilde{\mathcal{X}}_1 := g_1^* \mathcal{X}_1$ and $\tilde{\mathcal{X}}_2 := g_2^* \mathcal{X}_2$ in local coordinates around $g_i^{-1}(s_i)$ are locally pullbacks of \mathcal{X} by the map $t \mapsto t^{ab}$, in particular the formal neighbourhoods of X_i in $\tilde{\mathcal{X}}_i$ are isomorphic and the lemma follows by Theorem 4. \square

Theorem 6. *Let $f_1 : \mathcal{X} \rightarrow S_1$ and $f_2 : \mathcal{Y} \rightarrow S_2$ be two families of Calabi-Yau threefolds over algebraic curves, versal at a generic point and*

$$\varphi : X \dashrightarrow Y$$

a birational map between fibres $X = f_1^{-1}(s_1) \subset \mathcal{X}$ and $Y = f_2^{-1}(s_2) \subset \mathcal{Y}$, then the Picard-Fuchs operators of families \mathcal{X} and \mathcal{Y} are related.

Proof. By Proposition 3 there exist an étale neighbourhood (U, u_1) of the point $s_1 \in S_1$, a family $\tilde{\mathcal{Y}} \rightarrow U$ and a fiberwise birational map $\Phi : \mathcal{X}_U \dashrightarrow \tilde{\mathcal{Y}}$, where \mathcal{X}_U is the pull-back of \mathcal{X} to U . By Proposition 2 Picard-Fuchs operators of families \mathcal{X}_U and $\tilde{\mathcal{Y}}$ are strictly equivalent. The families $\tilde{\mathcal{Y}}$ and \mathcal{Y} are related by Lemma 5 and the theorem follows. \square

Corollary. *Let X be a Calabi-Yau threefold with $h^{1,2}(X) = 1$ and $\mathcal{X} \rightarrow S$ any non isotrivial family over an algebraic curve S , then the Picard-Fuchs operator, consider up to relation of relatedness, is a birational invariant of S .*

It is easy to see that under pull-back one can not get rid of a *MUM*, *K*, *C* or *A*-point. Only an *F*-point with equidistant exponents may turn into the non-singularity with exponents 0, 1, 2, 3. Note in particular that an operator with a *MUM*-point can not be related to an operator without a *MUM*-point, etc.

Our motivation to formulate the above theorem came from an attempt to distinguish double octics with the same Hodge numbers.

Example. The Calabi-Yau threefolds of arrangements **152** and **153** both have $h^{1,1} = 41$, so a priori could be birational. But no smooth fibre of arrangement **152** is birational to a smooth fibre of arrangement **153**. From the

analysis in next section we will see that the Picard-Fuchs operator of **152** has *KCCC* as singularities, whereas that of **153** has *ACCC* singularities, so these operators are not related.

We do not know of any other means of distinguishing members of these two families. We see that the Picard-Fuchs operator can be used as a powerful new birational invariant of a variety. We expect our theorem to be applicable to many other examples.

6. Orphans of order 2

It turns out that there are seven arrangements that lead to a second order operator. These are the arrangements

4, 13, 34, 72, 261, 264, 270.

The differential equation for each of these cases was computed; the results are recorded in Appendix B. All operators turn out to be of a very simple type, directly related to the Legendre differential equation, which is the hypergeometric equation

$$\Theta^2 - 16t(\Theta + 1/2)^2$$

and Riemann symbol

$$\left\{ \begin{array}{ccc} 0 & 1/16 & \infty \\ 0 & 0 & 1/2 \\ 0 & 0 & 1/2 \end{array} \right\}.$$

This also is the Picard-Fuchs operator of the elliptic surface with Kodaira fibres I_2, I_2, I_2^* . The differential equations in the cases **72** and **270** are a bit different.

At first sight it is very surprising to find a second order equation for such octic triple integrals. As explained in [40], the appearance of a certain *maximal automorphism* causes the Picard-Fuchs operator to be of order two. In [15] such maximal automorphisms were identified in five of the seven cases. For the remaining two cases **264** and **270** we do not have such a simple explanation for the appearance of a second order Picard-Fuchs equation.

A priori, there seem to be two different scenarios in which the Picard-Fuchs operator of a family can reduce to an operator of order two. It could happen that the $(1, 1, 1, 1)$ -VHS splits as sum into a (rigid) $(1, 0, 0, 1)$ Hodge structure and a variable $(0, 1, 1, 0)$, coming from an elliptic curve. Or it could

be that the $(1, 1, 1, 1)$ -VHS is a tensor product of a constant $(1, 0, 1)$ -Hodge structure with a variable $(1, 1)$ -VHS coming from a family of elliptic curves. This $1, 0, 1$ should be the transcendental part of H^2 of a K3-surface with Picard number 20.

In our situation it is always the second alternative that has to occur, as we are looking at the Picard-Fuchs operator for the period integrals of the holomorphic volume form ω , which in the first case would be constant.

It is of interest to identify the K3-surface in the geometry of the arrangement. For example, for the octic corresponding to arrangement **13**

$$Y_t : 0 = u^2 - xyz(x+y)(y+z)w(x-z-w)(x-z-tw)$$

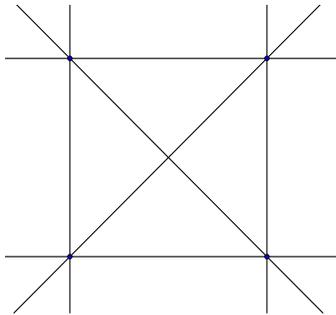
one can understand its relation to the K3-surface in the following way. Replacing u by $u/(x-z)$ we see the double octic is equal to the normalisation of the *double dectic*

$$u^2 = xyz(x+y)(y+z)(x-z)w(x-z-w)(x-z-tw)(x-z).$$

The first 6 factors now only depend on the variables x, y, z and determine a double sextic K3-surface S with equation

$$p^2 = xyz(x+y)(y+z)(x-z)$$

It is the famous *most algebraic K3-surface* [53], which comes with the weight 3 modular form named 16 in appendix A.



The last four factors only depend on w and $\xi := x-z$, with t as parameter and determine a double quartic family of elliptic curves E_t given by the equation

$$q^2 = w(\xi-w)(\xi-tw)\xi$$

By dividing out the involution ι induced by $p \mapsto -p, q \rightarrow -q$ acting on $S \times E_t$ we get back our double dectic, $u = pq$. Hence we see that our original

double octic is birational to

$$S \times E_t/\iota \longrightarrow Y_t$$

so that we see that Y_t is a simple instance of the *Borcea-Voisin construction*, see [11].

It turns out that, unexpectedly, in all cases except **270** we end up with the modular form 16 as constant factor.

In case **270** we get modular form 8, attached to the double sextic K3-surface

$$p^2 = xyz(x-y)(y+z)(x-z)$$

These two K3-surfaces appear in a nice pencil; in [21] one finds a very detailed account of their geometry and arithmetic.

7. Orphans of order 4

Of the 63 families it turns out that 18 are fourth order orphans. These are much more interesting and are collected in Appendix C. We sort them according to types of singularities that appear.

7.1. The two KKCC-operators

It turns out that there are two different but similar operators with two points of type K and two points of type C . In each case there is a pair of arrangements related to them.

The arrangements 33 and 70 with $h^{11} = 49$.

These two arrangements are birational, via the map

$$(x, y, z, v) \longmapsto (tyv, xy - yz, tzv - xz + z^2, x^2 - xz - txv).$$

(Here and below we will always refer to the equations of the arrangements given in Appendix C.)

The Picard-Fuchs operator for **33** has Riemann symbol

$$\left(\begin{array}{cccc} 0 & 1 & 2 & \infty \\ \hline 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1 & 1/2 \\ 1 & 1/2 & 1 & 3/2 \\ 1 & 2 & 2 & 3/2 \end{array} \right)$$

from which we see that 0 and ∞ are K -points, and 1 and 2 are C -points. The operator for **70** differs from it by $t \mapsto -t$. This operator was obtained in [13] by conifold expansion.

As explained above, to each rational K -point there is attached a weight three modular form and to each rational C -form a weight four modular form. It is well-known that these forms can be determined by counting points over finite fields and we will omit all details on their calculation.

We found it convenient to write the names of these forms *above* the corresponding points of the Riemann-symbol, as to form a *decorated Riemann-symbol*.

$$\left(\begin{array}{cccc} 8 & 32/2 & 8/1 & 16 \\ \hline 0 & 1 & 2 & \infty \\ \hline 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1 & 1/2 \\ 1 & 1/2 & 1 & 3/2 \\ 1 & 2 & 2 & 3/2 \end{array} \right)$$

(For the naming of the modular forms we are using the reader is referred to Appendix A.) We see that two different types of $K3$ -surfaces, corresponding to the forms 8 and 16 should appear in the semi-stable model of the singular fibres. This is in accordance with the fact that there is no symmetry that fixes the C -points and interchanges the K -points.

The arrangements 97 and 98 with $h^{11} = 45$.

This is the other pair of arrangements that also lead to an operator with two K and two C -points. The arrangements **97** and **98** are also birational, via the map

$$(x, y, z, v) \longmapsto ((tv - z - v)(x + y + z + v), txz, (-tv + z + v)y, (tv - z - v)(x + y))$$

The decorated Riemann symbol of the operator for **97** is

$$\left(\begin{array}{cccc} 32/1 & 8 & 8/1 & 8 \\ \hline 0 & -1 & -2 & \infty \\ \hline 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 1 & 1/2 \\ 1/2 & 1 & 1 & 3/2 \\ 1 & 1 & 2 & 3/2 \end{array} \right)$$

By shifting the exponents by $1/2$ in 0 and $-1/2$ in ∞ followed by a simple Möbius transformation one can transform it to operator **98**.

We again see two K points, but this time we find that at both of them the modular form is 8 , which suggests that the operator of **98** has a symmetry that interchanges the two K -points. Indeed, by shifting the finite C -point to the origin, one finds that the operator is symmetric under $t \mapsto -t$ and thus can be pulled-back by the squaring map from the nice operator

$$\mathcal{A} := \Theta^2(\Theta - 1)^2 + t\Theta^2(32\Theta^2 + 3) + 4t^2(4\Theta + 1)(2\Theta + 1)^2(4\Theta + 3)$$

which has extended Riemann symbol

$$\left(\begin{array}{ccc} 8 & 8/1 & 32/1 \\ \hline 0 & -1/16 & \infty \\ \hline 0 & 0 & 1/4 \\ 0 & 1/2 & 1/2 \\ 1 & 1/2 & 1/2 \\ 1 & 1 & 3/4 \end{array} \right)$$

The symmetry of the operator is also visible as a symmetry of the arrangements.

7.2. The two KCCC operators

It turns out that there are also two different operators with a single K point, but with three additional C -points. The first of these operators is related to two essentially different pairs of double octic arrangements, namely

The arrangements 35 and 71 with $h^{11} = 49$.

and

The arrangements 247 and 252 with $h^{11} = 37$.

As the Riemann-symbol suggests, the Picard-Fuchs operators for these four cases are related by a simple Möbius transformation and multiplication with an algebraic function and we will only analyse one of the cases.

The arrangements **247** and **252** are related by generically 2:1 correspondence ([15, Thm. 6.3]) that explain the equality of the Picard-Fuchs operators, but we were unable to find any correspondence between **35** and **71**. The coincidence of the Picard-Fuchs operators (up to transformation) strongly suggest that there exist a *correspondence* between **35** and **247**, but again we were unable to find it. This is an illustration of the power of Picard-Fuchs operators to make geometrical predictions.

The decorated Riemann symbol of **35** is

$$\left(\begin{array}{cccc} 8/1 & 8/1 & 8 & 8/1 \\ \hline -1 & 0 & 1 & \infty \\ \hline 0 & 0 & 0 & 1/2 \\ 1 & 1/2 & 0 & 1 \\ 1 & 1/2 & 1 & 1 \\ 2 & 1 & 1 & 3/2 \end{array} \right)$$

so at all C -points we find the modular form $8/1$.

Indeed, the operator of **35** has symmetries interchanging the C -points. If we shift the exponents at -1 by $1/2$, at ∞ by $-1/2$, and then bringing the K -point to 0 and the C -point at -1 to ∞ , we obtain an operator symmetric under $t \mapsto -t$, so it is seen to be pull-back by the squaring map of

$$\begin{aligned} \mathcal{B} := & \Theta^2 (2\Theta - 1)^2 + t(4\Theta^2 + 2\Theta + 1)(4\Theta + 1)^2 \\ & + t^2(4\Theta + 1)(4\Theta + 3)^2(4\Theta + 5) \end{aligned}$$

with decorated Riemann-Symbol

$$\left(\begin{array}{ccc} 8 & 8/1 & 8/1 \\ \hline 0 & -1/8 & \infty \\ \hline 0 & 0 & 1/4 \\ 0 & 1/2 & 3/4 \\ 1/2 & 1/2 & 3/4 \\ 1/2 & 1 & 5/4 \end{array} \right)$$

At both conifold points we have modular form $8/1$, and it turns out that there is a further symmetry in the operator that exchanges these and thus can be obtained as pull-back from yet another operator with three singular points:

$$\begin{aligned} & \Theta^2 (4\Theta - 1)^2 + 2t(8\Theta + 1)(32\Theta^3 + 28\Theta^2 + 19\Theta + 4) \\ & + t^2(8\Theta + 1)(8\Theta + 9)(8\Theta + 5)^2 \end{aligned}$$

Its decorated Riemann symbol is

$$\left(\begin{array}{ccc} 8 & ? & 8/1 \\ \hline 0 & -1/8 & \infty \\ \hline 0 & 0 & 1/8 \\ 0 & 1/2 & 5/8 \\ 1/4 & 1 & 5/8 \\ 1/4 & 3/2 & 9/8 \end{array} \right)$$

Note that now there appears a singularity at $-1/8$ with monodromy of order 2. So the Calabi-Yau threefold appearing at this point is special, but the ? indicates that we were not able yet to identify any modular forms.

The arrangements 152 and 198 with $h^{11} = 41$.

These two arrangement give rise to another operator with one K and three C -points. The coincidence of Hodge numbers and Picard-Fuchs operator (up to equivalence) suggest, that the two arrangements are birational. Again, we were unable to find the map.

The decorated Riemann symbol of **152** is

$$\left(\begin{array}{cccc} 8 & 8/1 & 32/1 & 8/1 \\ \hline -1 & 0 & 1 & \infty \\ \hline 0 & 0 & 0 & 1/2 \\ 1/2 & 1/2 & 0 & 1 \\ 1/2 & 1/2 & 2 & 1 \\ 1 & 1 & 2 & 3/2 \end{array} \right)$$

The operator of **198** is equivalent to it: by shifting the exponents at -1 by $1/2$ and at ∞ by $-1/2$.

The appearance of the forms $8/1$ at both 0 and ∞ suggests that there is a symmetry interchanging these points. And indeed, it turns out that the operator for **152** is also a pull-back from the operator \mathcal{A} !

7.3. The ACCC-operator

The arrangements 153, 197 with $h^{11} = 41$

The arrangements **153** and **197** are birational, via the map

$$(x, y, z, v) \mapsto ((xt + vt - y)tv, (xt + vt - y - z)y, (x + v)tz, (xt + vt - y)tx)$$

The decorated Riemann symbol for **153** is:

$$\left(\begin{array}{cccc} 32/1 & 32/2 & 8/1 & 32/2 \\ \hline 0 & -1 & -2 & \infty \\ \hline 0 & 0 & 0 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & 3/2 & 1 \\ 1 & 1 & 2 & 3/2 \end{array} \right)$$

The point $t = -2$ is very remarkable. By expanding the solutions around the singular point -2 one sees that the monodromy is of order two; no logarithmic terms arise. As a result, the corresponding limiting MHS remains pure of weight three. On the other hand, the arrangement at $t = -2$ specialises to the rigid arrangement **93** and the corresponding double octic has a rigid Calabi-Yau (with modular form $8/1$) as resolution. This implies that any semi-stable fibre at $t = -2$ needs to have further components to account for the change in cohomology between special fibre at -2 and general fibre. We note that the theorem of KULIKOV implies that a similar phenomenon can not happen for K3-surfaces. This and other examples will be studied in more detail in future paper.

As suggested by the modular forms, there could be a symmetry interchanging the two $32/2$ points. By first shifting exponents by $1/2$ at -2 and $-1/2$ at ∞ and then bringing -2 to ∞ , the operator is pulled back from the nice operator with three singular points:

$$\begin{aligned} \mathcal{C} := & \Theta(4\Theta - 1)^2(2\Theta - 1) - t(4\Theta + 1)^2(4\Theta^2 + 2\Theta + 1) \\ & + t^2(4\Theta + 1)(2\Theta + 1)(4\Theta + 5)(\Theta + 1) \end{aligned}$$

with extended Riemann symbol

$$\left(\begin{array}{ccc} 32/1 & 32/2 & 8/1 \\ \hline 0 & 1 & \infty \\ \hline 0 & 0 & 1/4 \\ 1/4 & 1/2 & 1/2 \\ 1/4 & 1/2 & 1 \\ 1/2 & 1 & 5/4 \end{array} \right).$$

7.4. The KCCCC-operator

The arrangement **243** ($h^{11} = 39$) leads to the rather complicated operator

$$\Theta(\Theta - 2)(\Theta - 1)^2 - \frac{1}{6}t\Theta(\Theta - 1)(19\Theta^2 - 19\Theta + 9) + \frac{1}{3}t^2\Theta^2(11\Theta^2 + 4) \\ - \frac{1}{24}t^3(11\Theta^2 + 11\Theta + 5)(2\Theta + 1)^2 + \frac{1}{48}t^4(2\Theta + 3)^2(2\Theta + 1)^2$$

Its decorated Riemann symbol is

$$\left(\begin{array}{ccccc} 12/1 & 32/2 & 6/1 & 8/1 & 8 \\ \hline 0 & 1 & \frac{3}{2} & 2 & \infty \\ \hline 0 & 0 & 0 & 0 & 1/2 \\ 1 & 1/2 & 1 & 1 & 1/2 \\ 1 & 1/2 & 1 & 1 & 3/2 \\ 2 & 1 & 2 & 2 & 3/2 \end{array} \right)$$

We claim it can not be simplified further, as the modular forms at the cusps are all different.

7.5. The ACCCK-operator

By [15, Thm. 6.2] arrangements **250** and **258** define birational families of Calabi-Yau threefolds with $h^{11} = 37$. The decorated Riemann-symbol of **250** is

$$\left(\begin{array}{ccccc} 6/1 & 8/1 & h & 8/1 & 6/1 & 8 \\ \hline -2 & -1 & -1/2 & 0 & 1 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1 & 1/2 & 1 & 1/2 & 1 & 1/2 \\ 1 & 1/2 & 3 & 1/2 & 1 & 3/2 \\ 2 & 1 & 4 & 1 & 2 & 3/2 \end{array} \right)$$

Very remarkably, at the apparent singularity at the point $-1/2$ there appears the Hilbert-modular form h of level $6\sqrt{2}$ and weight $(4, 2)$. When we shift this apparent singularity to the origin, we obtain an operator that is symmetric with respect to the involution $t \mapsto -t$. But there is no obvious corresponding symmetry in the family. Counting points, it appears that the number of points of the fibres at t and $-t$ are equal or opposite mod p , according to the quadratic character $\left(\frac{2}{p}\right)$. In fact the transformation

$$\begin{pmatrix} x \\ y \\ z \\ v \end{pmatrix} \mapsto \begin{pmatrix} 2(-y-z+v)x(x+v+y+z)((t+1/2)y-v-x-z) \\ 4(x+v+y+z)(-1/2z^2+(v/2-x/2-y)z-1/2y^2+(v/2-x/2)y+vx)z \\ (y^2+(-v+x+2z)y+z^2+(x-v)z-2vx)(x+v+y+z)((t+1/2)y-v-x+z) \\ (y+z)(v-x-y-z)^2(ty-v-x+y/2-z) \end{pmatrix}$$

gives a correspondance between a fiber of the family and the quadratic twist by 2 of the opposite fiber. In particular, the modular forms labelled 8/1 actually occuring differ by this character. From this state of affairs it seems natural that at the symmetry point the Hilbert modular form for $\sqrt{2}$ appears. We plan study this example more carefully in a future paper.

We remark that although the modular forms at corresponding fibres 0 and -1 are in both cases 8/1, they correspond to two non-birational rigid Calabi-Yau configuration **69** ($h^{11} = 50$) and **93** ($h^{11} = 46$).

Similarly, at 1 and -2 we have modular form 6/1, but rigid Calabi-Yau configurations **245** ($h^{11} = 38$) resp. **240** ($h^{11} = 40$). So it seems improbable that there is a birational map relating the fibre at t to the fibre at $-1/2 - t$. Geometrically, the symmetry of the Picard-Fuchs operator is surprising.

Using the symmetry we see that the operator is pulled back from a simpler operator with the following Riemann symbol:

$$\left\{ \begin{array}{cccc} h & 8/1 & 6/1 & \infty \\ \hline 0 & 1 & 9 & \infty \\ \hline 0 & 0 & 0 & 1/4 \\ 1/2 & 1/2 & 1 & 1/4 \\ 3/2 & 1/2 & 1 & 3/4 \\ 2 & 1 & 2 & 3/4 \end{array} \right\}$$

7.6. The KCCCC-operator

Arrangement 248 with $h^{12} = 37$.

Decorated Riemann symbol

$$\left\{ \begin{array}{cccccc} 12/1 & 6/1 & 16 & 6/1 & 12/1 & 32/1 \\ \hline -2 & -3/2 & -1 & -1/2 & 0 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 3/2 \end{array} \right\}$$

There is a symmetry pairing the points with same modular form and fixing the points belonging to the modular forms 16 and 32/1. First shift the K -point to 0, then the operator is seen to be pull-back via quadratic map from an operator with Riemann symbol

$$\left(\begin{array}{cccc} 16 & 12/1 & 6/1 & 32/1 \\ \hline 0 & 1 & 1/4 & \infty \\ \hline 0 & 0 & 0 & 1/4 \\ 0 & 1 & 1 & 1/2 \\ 1 & 1 & 1 & 1/2 \\ 1 & 2 & 2 & 3/4 \end{array} \right).$$

This symmetry, after having discovered it from the operator, can be seen in the arrangement.

7.7. The AACCCCC-operator

Arrangements 266 and 273 have both $h^{11} = 37$ and have rather complicated but *identical* Picard-Fuchs equations, with Riemann symbol

$$\left(\begin{array}{ccccccccc} 6/1 & 32/1 & 6/1 & 32/1 & 6/1 & ? & ? & 32/1 \\ \hline -1 & -1/2 & -1/4 & 0 & 1/2 & (-1 + \sqrt{-3})/4 & (-1 - \sqrt{-3})/4 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1 & 1/2 & 1 & 1/2 & 1 & 1 & 1 & 1 \\ 1 & 1/2 & 1 & 1/2 & 1 & 3 & 3 & 1 \\ 2 & 1 & 2 & 1 & 2 & 4 & 4 & 3/2 \end{array} \right)$$

Arrangement **266** and **273** both have 8 quadruple points and no five-fold point or triple lines. The difference between the two is rather subtle; **266** contains six planes which are in general position, which is not the case for **273**. In **266** there are six of the quadruple points in a plane, which is not the case for **273**, etc. So clearly the two configurations are projectively very different. On the other hand, they have the same Hodge number $h^{11} = 37$, and the equality of their Picard-Fuchs operators clearly suggest the varieties belonging to the two arrangements are birational, but were unable to find any transformation.

The operator can be reduced to one with three singular points via the following steps

- Make the exponents at all 32/1-points equal to $(0, 1, 1, 2)$, by exponent-shift.

- Translate the A -points to 0 and ∞ . Of course, the resulting operator has coefficients in $\mathbb{Q}(\sqrt{-3})$.
- Make a pull-back of order three, i.e. we use t^3 as new coordinate.
- The operator has now four singular points, where the points at 0 and ∞ have exponents 0, $1/3$, 1 , $4/3$. Bring the two other points to 0 and ∞ .
- The result is an operator invariant under $t \mapsto -t$. Make a quadratic pull-back and bring the A point to the origin.
- The result is, after a scaling of the coordinate, the following operator

$$4\Theta(3\Theta - 1)(\Theta - 1)(3\Theta - 4) + 3t\Theta(3\Theta - 1)(288\Theta^2 - 96\Theta + 35) \\ + 36t^2(12\Theta + 1)(3\Theta + 1)^2(12\Theta + 7)$$

with decorated Riemann symbol

$$\left(\begin{array}{ccc|ccc} ? & 6/1 & 32/1 & & & \\ \hline 0 & -1/36 & \infty & & & \\ \hline 0 & 0 & 1/12 & & & \\ 1/3 & 1/2 & 1/3 & & & \\ 1 & 1/2 & 1/3 & & & \\ 4/3 & 1 & 7/12 & & & \\ \hline F & C & C & & & \end{array} \right)$$

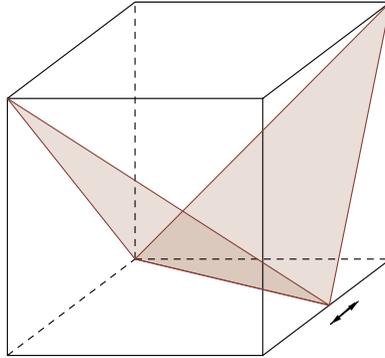
We were informed by Stefan Reiter that the operators with three singular points that appeared in 7.1, 7.2, 7.3 and 7.7, up to shift of exponents, are symplectically rigid and can be constructed using convolutions as in [7], but were not included in that paper because they do not have a MUM point and hence do not classify as a Calabi-Yau operator.

8. Concluding remarks and questions

As for families of $K3$ -surfaces or other varieties, there is nothing special about families of Calabi-Yau three-folds that avoid having degenerations with maximal unipotent monodromy points and still have monodromy Zariski dense in $Sp_4(\mathbb{C})$. One may ask further questions about the possible distribution of types of singularities. For example, the examples with second order Picard-Fuchs operators give variations which have only K -points. Do there exist families with only K -points and Picard-Fuchs operator of order four? Similarly, do there exist families with only C -points? It seems that only the current general lack of examples is the reason for this kind of ignorance.

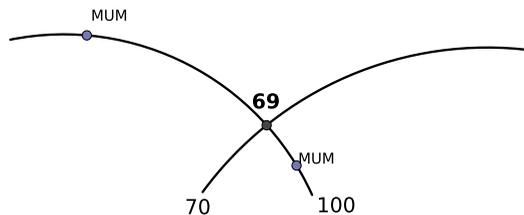
As for mirror symmetry, the SYZ-approach via dual torus fibrations (T-duality), [49], does in no way presupposes the presence of a MUM-point in the moduli space. The point is rather that the absence of a MUM-point does not give a clue where to look for an appropriate torus fibration. However, there is the well-known idea, going back to MILES REID [39], that the different families of Calabi-Yau threefolds may all be connected via geometrical transitions. The most prominent such transition is the *conifold transition*, where we contract some lines on a Calabi-Yau threefold to form nodes, and smooth these out to produce another Calabi-Yau threefold with different Hodge numbers. As has been suggested by MORRISON, [37], mirror symmetry can, in some sense, be prolonged over such transitions, and in the case of the above mentioned conifold transition, the mirror symmetric process is that of nodal degeneration, followed by a crepant resolution. So if we connect an orphan family to another family that has a MUM-point via a transition, one is tempted to try construct a mirror manifold for orphans using this transition to a family with a MUM-point.

We have seen in section 1 that the rigid Calabi-Yau from arrangement **69** appears as a member of the orphan family **70**. But **69** also is member of the family **100**, which contains even two MUM-points.



Family of arrangements 100

So we have the following picture



In the second part of the paper we will report on the double octics families that contain a MUM-point.

Appendix A

We will encounter certain modular forms over and over again. For the convenience of the reader we list here the first few Fourier coefficients of these forms.

Weight two modular form: Associated to the elliptic curve

$$y^2 = x^3 - x$$

of conductor $2^5 = 32$ there is the unique cusp form, [25]:

$$f_{32} := q \prod_{n=1}^{\infty} (1 - q^{4n})^2 (1 - q^{8n})^2 \in S_2(\Gamma_0(32))$$

Name	2	3	5	7	11	13	17	19	23	29
f_{32}	0	0	-2	0	0	6	2	0	0	-10

Weight 3 modular forms: The weight three cusp forms for $\Gamma_0(N)$ all will appear with a non-trivial character that will play no role for us. The forms of level 8 and 16 are uniquely determined by there level, which we use to name them.

Name	2	3	5	7	11	13	17	19	23	<i>CM - type</i>
16	0	0	-6	0	0	10	-30	0	0	$\mathbb{Q}(\sqrt{-1})$
8	-2	-2	0	0	14	0	2	-34	0	$\mathbb{Q}(\sqrt{-2})$

These two forms are η -products:

$$8 := q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{2n}) (1 - q^{4n}) (1 - q^{8n})^2$$

$$16 := q \prod_{n=1}^{\infty} (1 - q^{4n})^6$$

The Galois representation associated to the form 8 is the tensor square of the Galois action associated to the form f_{32} . For more information on weight 3 forms we refer to [44].

Weight 4 cusp forms

For the weight 4 cusp forms for $\Gamma_0(N)$ we will use the notation used in the book of MEYER [35].

<i>Name</i>	2	3	5	7	11	13	17	19	23
6/1	-2	-3	6	-16	12	38	-126	20	168
8/1	0	-4	-2	24	-44	22	50	44	-56
12/1	0	3	-18	8	36	-10	18	-100	72
32/1	0	0	22	0	0	-18	-94	0	0
32/2	0	8	-10	16	-40	-50	-30	40	48

The first two of these forms are also nice η -products:

$$6/1 = q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{2n})^2 (1 - q^{3n})^2 (1 - q^{6n})^2$$

$$8/1 = q \prod_{n=1}^{\infty} (1 - q^{2n})^4 (1 - q^{4n})^4$$

The Galois representation associated to the form 32/1 is the tensor cube of the Galois action associated to the form f_{32} .

There is also one Hilbert modular form h for the field $\mathbb{Q}(\sqrt{2})$ that plays a role. It is the unique cuspform of weight $(4, 2)$ and level $6\sqrt{2}$; its coefficients for the first inert primes are:

<i>Name</i>	2	3	5	7	11	13	17	19	23
h	0	9	10	$4\sqrt{2} + 16$	-726	2938	$16\sqrt{2} - 62$	6650	$-8\sqrt{2} + 40$

Appendix B

4. $xyzv(x+y)(x+ty+tz-v)(x+y+tz-v)(y+z)$

$$\Theta^2 - t\left(\Theta + \frac{1}{2}\right)^2$$

$$\left\{ \begin{array}{ccc} 1 & 0 & \infty \\ 0 & 0 & 1/2 \\ 0 & 0 & 1/2 \end{array} \right\}$$

13. $xyzv(z+y)(x-z-v)(x+y)(x-z+tv)$

$$\Theta^2 + t\left(\Theta + \frac{1}{2}\right)^2$$

$$\left\{ \begin{array}{cccc} 0 & -1 & \infty & \\ 0 & 0 & 1/2 & \\ 0 & 0 & 1/2 & \end{array} \right\}$$

34. $xyzv(x+y)(x+z)(x+y+z+v)(y-z+tv)$
 $\Theta^2 - t^2(\Theta + \frac{1}{2})^2$

$$\left\{ \begin{array}{cccc} 1 & 0 & -1 & \infty \\ 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 1/2 \end{array} \right\}$$

72. $xyzv(x-y-v)(x+y+z)(y+tz+tv)(y+z+v)$
 $\Theta^2 + t(-3\Theta^2 - 2\Theta - \frac{1}{2}) + t^2(\Theta + 1)(2\Theta + 1)$

$$\left\{ \begin{array}{cccc} 1 & 1/2 & 0 & \infty \\ 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1 \end{array} \right\}$$

261. $xyzv(x-y-z+v)(x+y+z+v)(x-y+tz-tv)$
 $\times (x+y+tz+tv)$
 $\Theta^2 - t^2(\Theta + 1)^2$

$$\left\{ \begin{array}{cccc} 1 & 0 & -1 & \infty \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array} \right\}$$

264. $xyzv(x + (2-t)v + 2y - (2-t)z)(-x - y + 2z - (2-t)v)$
 $\times (x + y + tz)(y + 2 - 2z)$
 $\Theta^2 - 1/4t^2(\Theta + 1)^2$

$$\left\{ \begin{array}{cccc} 2 & 0 & -2 & \infty \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array} \right\}$$

270. $xyzv(x+y+z)(y+z+v)(xt-2y+tz+tv)(-x-2y+tz-v)$
 $\Theta^2 + t(\frac{3}{2}\Theta^2 + \frac{3}{2}\Theta + \frac{1}{2}) + \frac{1}{2}t^2(\Theta + 1)^2$

$$\left\{ \begin{array}{cccc} 0 & -1 & -2 & \infty \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array} \right\}$$

Appendix C

33. $xyzv(x+y)(y+z)(x-z+v)(x-y-z+tv)$
 $\Theta^2(\Theta - 1)^2 - \frac{1}{8}t\Theta^2(20\Theta^2 + 3) + \frac{1}{16}t^2(8\Theta^2 + 8\Theta + 3)(2\Theta + 1)^2$

$$-\frac{1}{32}t^3(2\Theta+3)^2(2\Theta+1)^2$$

$$\left\{ \begin{array}{cccc} 0 & 1 & 2 & \infty \\ 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1 & 1/2 \\ 1 & 1/2 & 1 & 3/2 \\ 1 & 1 & 2 & 3/2 \end{array} \right\}$$

35. $xyz(x-v)(y-v)(z-v)(x-y)(x+ty+(1-t)z-v)$
 $\Theta(\Theta-1)(\Theta-\frac{1}{2})^2 - \frac{1}{4}t\Theta^2(4\Theta^2+3) - \frac{1}{4}t^2(\Theta^2+\Theta+1)$
 $\times (2\Theta+1)^2 + \frac{1}{4}t^3(2\Theta+3)(2\Theta+1)(\Theta+1)^2$

$$\left\{ \begin{array}{cccc} 1 & 0 & -1 & \infty \\ 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1 & 1 \\ 1 & 1/2 & 1 & 1 \\ 1 & 1 & 2 & 3/2 \end{array} \right\}$$

70. $yxzv(x+ty)(y-z-v)(x-y-v)(x-y+z)$
 $\Theta^2(\Theta-1)^2 + \frac{1}{8}t\Theta^2(20\Theta^2+3) + \frac{1}{16}t^2(8\Theta^2+8\Theta+3)(2\Theta+1)^2$
 $+ \frac{1}{32}t^3(2\Theta+3)^2(2\Theta+1)^2$

$$\left\{ \begin{array}{cccc} 0 & -1 & -2 & \infty \\ 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1 & 1/2 \\ 1 & 1/2 & 1 & 3/2 \\ 1 & 1 & 2 & 3/2 \end{array} \right\}$$

71. $xyzv(x+y)(x+y+z+v)(ty-z-v)(-x+ty-z)$
 $\Theta(\Theta-1)(\Theta-\frac{1}{2})^2 + t\Theta^2(4\Theta^2+1) + \frac{1}{16}t^2(20\Theta^2+20\Theta+9)$
 $\times (2\Theta+1)^2 + \frac{1}{8}t^3(2\Theta+3)^2(2\Theta+1)^2$

$$\left\{ \begin{array}{cccc} 0 & -1/2 & -1 & \infty \\ 0 & 0 & 0 & 1/2 \\ 1/2 & 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 & 3/2 \\ 1 & 2 & 1 & 3/2 \end{array} \right\}$$

97. $xyzv(x+y)(x+y+z+v)(-x+tz-v)(y-v-z)$
 $\Theta(\Theta-1)(\Theta-\frac{1}{2})^2 + \frac{1}{8}t\Theta^2(20\Theta^2+7)$
 $+ \frac{1}{32}t^2(16\Theta^2+16\Theta+7)(2\Theta+1)^2 + \frac{1}{32}t^3(2\Theta+3)^2(2\Theta+1)^2$

$$\left\{ \begin{array}{cccc} 0 & -1 & -2 & \infty \\ \hline 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 1 & 1/2 \\ 1/2 & 1 & 1 & 3/2 \\ 1 & 1 & 2 & 3/2 \end{array} \right\}$$

98. $xyzv(x+z-v)(x+z+y)(y+z+v)(y+tz+tv)$
 $\Theta^2(\Theta-1)^2 - \frac{1}{4}t\Theta^2(16\Theta^2+3) + \frac{1}{4}t^2(5\Theta^2+5\Theta+3)(2\Theta+1)^2$
 $-\frac{1}{2}t^3(2\Theta+3)(2\Theta+1)(\Theta+1)^2$

$$\left\{ \begin{array}{cccc} 1 & 1/2 & 0 & \infty \\ \hline 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 3/2 \end{array} \right\}$$

152. $xyzv(x+v-y-z)(x+y+z+v)(x-y+tz-tv)(y+v)$
 $\Theta(\Theta-1)(\Theta-\frac{1}{2})^2 + \frac{1}{2}t\Theta(2\Theta^3-8\Theta^2+6\Theta-1)$
 $+t^2(-2\Theta^4-4\Theta^3-\frac{11}{4}\Theta^2-\frac{17}{4}\Theta-\frac{11}{16})$
 $+t^3(-2\Theta^4+\frac{1}{4}\Theta^2+\frac{7}{2}\Theta+\frac{9}{8})$
 $+\frac{1}{16}t^4(2\Theta+1)(8\Theta^3+44\Theta^2+62\Theta+25)$
 $+\frac{1}{4}t^5(2\Theta+3)(2\Theta+1)(\Theta+1)^2$

$$\left\{ \begin{array}{cccc} 1 & 0 & -1 & \infty \\ \hline 0 & 0 & 0 & 1/2 \\ 1/2 & 1/2 & 0 & 1 \\ 1/2 & 1/2 & 2 & 1 \\ 1 & 1 & 2 & 3/2 \end{array} \right\}$$

153. $xyzv(x+z+y)(-x+ty-v)(-x+ty-z-v)(y+z+v)$
 $\Theta(\Theta-1)(\Theta-\frac{1}{2})^2 + \frac{1}{8}t\Theta(28\Theta^3-16\Theta^2+17\Theta-2)$
 $+t^2(\frac{19}{4}\Theta^4+\frac{7}{2}\Theta^3+\frac{39}{8}\Theta^2+\frac{13}{8}\Theta+\frac{19}{64})$
 $+t^3(\frac{25}{8}\Theta^4+6\Theta^3+\frac{109}{16}\Theta^2+\frac{7}{2}\Theta+\frac{89}{128})$
 $+\frac{1}{64}t^4(2\Theta+1)(32\Theta^3+80\Theta^2+82\Theta+29)$
 $+\frac{1}{32}t^5(2\Theta+3)(2\Theta+1)(\Theta+1)^2$

$$\left\{ \begin{array}{cccc} 0 & -1 & -2 & \infty \\ \hline 0 & 0 & 0 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & 3/2 & 1 \\ 1 & 1 & 2 & 3/2 \end{array} \right\}$$

197. $xyzv(x-y-z+v)(x+tz+v)(x+ty+tz)(ty+tz+v)$
 $\Theta^2(\Theta - \frac{1}{2})(\Theta + \frac{1}{2}) + \frac{1}{8}t(2\Theta + 1)(32\Theta^3 + 16\Theta^2 + 18\Theta + 5)$
 $+t^2(25\Theta^4 + 52\Theta^3 + \frac{121}{2}\Theta^2 + 37\Theta + \frac{145}{16})$
 $+t^3(38\Theta^4 + 124\Theta^3 + 183\Theta^2 + 133\Theta + \frac{307}{8})$
 $+t^4(\Theta + 1)(28\Theta^3 + 100\Theta^2 + 133\Theta + 63)$
 $+2t^5(\Theta + 2)(\Theta + 1)(2\Theta + 3)^2$

$$\left\{ \begin{array}{cccc} 0 & -1/2 & -1 & \infty \\ \hline -1/2 & 0 & 0 & 1 \\ 0 & 1/2 & 1/2 & 3/2 \\ 0 & 3/2 & 1/2 & 3/2 \\ 1/2 & 2 & 1 & 2 \end{array} \right\}$$

198. $xyzv(x-y-v)(x+y+z)(x-y-z+tx)(y+z+v)$
 $\Theta(\Theta - 1)(\Theta - \frac{1}{2})^2 + \frac{1}{8}t\Theta^2(24\Theta^2 + 5)$
 $+t^2(\frac{13}{4}\Theta^4 + \frac{13}{2}\Theta^3 + \frac{81}{16}\Theta^2 + \frac{29}{16}\Theta + \frac{5}{16})$
 $+t^3(\frac{3}{2}\Theta^4 + 6\Theta^3 + 8\Theta^2 + 4\Theta + \frac{25}{32}) + \frac{1}{64}t^4(2\Theta + 5)^2(2\Theta + 1)^2$

$$\left\{ \begin{array}{cccc} 0 & -1 & -2 & \infty \\ \hline 0 & 0 & 0 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & 5/2 \\ 1 & 1 & 1 & 5/2 \end{array} \right\}$$

243. $xyzv(x+y+v)(x+y+z)(x+ty+z+v)(y+z+v)$
 $\Theta(\Theta - 2)(\Theta - 1)^2 - \frac{1}{6}t\Theta(\Theta - 1)(19\Theta^2 - 19\Theta + 9)$
 $+ \frac{1}{3}t^2\Theta^2(11\Theta^2 + 4) - \frac{1}{24}t^3(11\Theta^2 + 11\Theta + 5)(2\Theta + 1)^2$
 $+ \frac{1}{48}t^4(2\Theta + 3)^2(2\Theta + 1)^2$

$$\left\{ \begin{array}{ccccc} 2 & 3/2 & 1 & 0 & \infty \\ \hline 0 & 0 & 0 & 0 & 1/2 \\ 1 & 1 & 1/2 & 1 & 1/2 \\ 1 & 1 & 1/2 & 1 & 3/2 \\ 2 & 2 & 1 & 2 & 3/2 \end{array} \right\}$$

$$\begin{aligned}
\mathbf{247.} \quad & xyzv(x-y-v)(x+y+z)(-x+tz-tv)(y+z+v) \\
& \Theta^2(\Theta-1)^2 + t\Theta^2(5\Theta^2+1) + t^2(2\Theta^2+2\Theta+1)(2\Theta+1)^2 \\
& + t^3(2\Theta+3)(2\Theta+1)(\Theta+1)^2 \\
& \left\{ \begin{array}{cccc} 0 & -1/2 & -1 & \infty \\ 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1 & 1 \\ 1 & 1/2 & 1 & 1 \\ 1 & 1 & 2 & 3/2 \end{array} \right\}
\end{aligned}$$

$$\begin{aligned}
\mathbf{248.} \quad & xyzv(x+z+v)(x+y+z)(x+(y+1)y-tz+v)(y-z-v) \\
& \Theta(\Theta-2)(\Theta-1)^2 + \frac{1}{6}t\Theta(\Theta-1)(37\Theta^2-61\Theta+36) \\
& + \frac{1}{6}t^2\Theta(91\Theta^3-124\Theta^2+121\Theta-36) \\
& + t^3\left(\frac{115}{6}\Theta^4 - \frac{5}{3}\Theta^3 + \frac{107}{6}\Theta^2 + \frac{2}{3}\Theta + \frac{1}{2}\right) \\
& + t^4\left(\frac{79}{6}\Theta^4 + 16\Theta^3 + \frac{113}{6}\Theta^2 + 8\Theta + \frac{3}{2}\right) \\
& + \frac{1}{6}t^5(2\Theta+1)(14\Theta^3+29\Theta^2+27\Theta+9) \\
& + \frac{1}{6}t^6(2\Theta+3)(2\Theta+1)(\Theta+1)^2 \\
& \left\{ \begin{array}{cccccc} 0 & -1/2 & -1 & -3/2 & -2 & \infty \\ 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 3/2 \end{array} \right\}
\end{aligned}$$

$$\begin{aligned}
\mathbf{250.} \quad & xyzv(x+y+z)(x+ty-z+v)(x+z+v)(y+z-v) \\
& \Theta(\Theta-1)(\Theta-\frac{1}{2})^2 + \frac{1}{8}t\Theta(44\Theta^3-96\Theta^2+65\Theta-12) \\
& + t^2\left(\frac{19}{2}\Theta^4 - 23\Theta^3 + \frac{131}{8}\Theta^2 - \frac{47}{8}\Theta - \frac{1}{4}\right) \\
& + t^3\left(\frac{5}{2}\Theta^4 - 20\Theta^3 - \frac{23}{4}\Theta - \frac{17}{32}\right) \\
& - \frac{1}{32}t^4(68\Theta^2+100\Theta+53)(2\Theta+1)^2 \\
& - \frac{1}{4}t^5(8\Theta^2+14\Theta+9)(2\Theta+1)^2 - \frac{1}{8}t^6(2\Theta+3)^2(2\Theta+1)^2 \\
& \left\{ \begin{array}{cccccc} 1 & 0 & -1/2 & -1 & -2 & \infty \\ 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1 & 1/2 & 1 & 1/2 & 1 & 1/2 \\ 1 & 1/2 & 3 & 1/2 & 1 & 3/2 \\ 2 & 1 & 4 & 1 & 2 & 3/2 \end{array} \right\}
\end{aligned}$$

$$\begin{aligned}
\mathbf{252.} \quad & xyzv(x+y+v)(x+y+z)(-x+tz+v)(-x-2y+tz-v) \\
& \Theta^2(\Theta-1)^2 + \frac{1}{2}t\Theta^2(5\Theta^2+1) + \frac{1}{4}t^2(2\Theta^2+2\Theta+1)(2\Theta+1)^2
\end{aligned}$$

$$+\frac{1}{8}t^3(2\Theta+3)(2\Theta+1)(\Theta+1)^2$$

$$\left\{ \begin{array}{cccc} 0 & -1 & -2 & \infty \\ 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1 & 1 \\ 1 & 1/2 & 1 & 1 \\ 1 & 1 & 2 & 3/2 \end{array} \right\}$$

258. $xyzv(x-y+2z-2v)(x-y+z-v)(x+ty+z+tv)$
 $\times(y-z+2v)$

$$\Theta(\Theta-1)(\Theta-\frac{1}{2})^2 + \frac{1}{8}t\Theta(20\Theta^3+48\Theta^2-21\Theta+6)$$

$$+t^2(-\Theta^4+19\Theta^3+\frac{39}{2}\Theta^2+\frac{47}{8}\Theta+\frac{11}{8})$$

$$+t^3(-5\Theta^4-5\Theta^3+\frac{61}{2}\Theta^2+\frac{127}{8}\Theta+\frac{37}{8})$$

$$+t^4(-\Theta^4-21\Theta^3-\frac{21}{2}\Theta^2-\frac{9}{8}\Theta+\frac{7}{8})$$

$$+\frac{1}{8}t^5(2\Theta+1)(10\Theta^3-9\Theta^2-27\Theta-13)$$

$$+\frac{1}{4}t^6(2\Theta+3)(2\Theta+1)(\Theta+1)^2$$

$$\left\{ \begin{array}{cccccc} 1 & 0 & -1/2 & -1 & -2 & \infty \\ 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1 & 1/2 & 1 & 0 & 1 & 1 \\ 3 & 1/2 & 1 & 1 & 1 & 1 \\ 4 & 1 & 2 & 1 & 2 & 3/2 \end{array} \right\}$$

266. $xyzv(2x+y+2v)(x+(t+1)y-z+v)(x+ty+z)(y-2z+2v)$

$$\Theta(\Theta-1)(\Theta-\frac{1}{2})^2 + \frac{1}{4}t\Theta(44\Theta^3-48\Theta^2+37\Theta-6)$$

$$+t^2(50\Theta^4-56\Theta^3+40\Theta^2-\frac{5}{2}\Theta+\frac{3}{8})$$

$$+t^3(120\Theta^4-288\Theta^3-75\Theta^2-105\Theta-21)$$

$$+t^4(112\Theta^4-1008\Theta^3-718\Theta^2-720\Theta-\frac{303}{2})$$

$$+t^5(-224\Theta^4-2464\Theta^3-1924\Theta^2-1628\Theta-324)$$

$$+t^6(-960\Theta^4-4224\Theta^3-4296\Theta^2-2448\Theta-450)$$

$$+t^7(-1600\Theta^4-4992\Theta^3-6368\Theta^2-3504\Theta-696)$$

$$-32t^8(2\Theta+1)(22\Theta^3+57\Theta^2+59\Theta+21)$$

$$-128t^9(2\Theta+3)(2\Theta+1)(\Theta+1)^2$$

$$\left\{ \begin{array}{cccccccc} 1/2 & 0 & -1/4 & -1/2 & -1 & (-1+\sqrt{-3})/4 & (-1-\sqrt{-3})/4 & \infty \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1 & 1/2 & 1 & 1/2 & 1 & 1 & 1 & 1 \\ 1 & 1/2 & 1 & 1/2 & 1 & 3 & 3 & 1 \\ 2 & 1 & 2 & 1 & 2 & 4 & 4 & 3/2 \end{array} \right\}$$

$$\begin{aligned}
\mathbf{273.} \quad & xyzv(x+y+z)(2x-2z-v)(x+2ty-z+tv)(2y+2z+v) \\
& \Theta(\Theta-1)\left(\Theta-\frac{1}{2}\right)^2 + \frac{1}{4}t\Theta(44\Theta^3-48\Theta^2+37\Theta-6) \\
& + t^2(50\Theta^4-56\Theta^3+40\Theta^2-\frac{5}{2}\Theta+\frac{3}{8}) \\
& + t^3(120\Theta^4-288\Theta^3-75\Theta^2-105\Theta-21) \\
& + t^4(112\Theta^4-1008\Theta^3-718\Theta^2-720\Theta-\frac{303}{2}) \\
& + t^5(-224\Theta^4-2464\Theta^3-1924\Theta^2-1628\Theta-324) \\
& + t^6(-960\Theta^4-4224\Theta^3-4296\Theta^2-2448\Theta-450) \\
& + t^7(-1600\Theta^4-4992\Theta^3-6368\Theta^2-3504\Theta-696) \\
& - 32t^8(2\Theta+1)(22\Theta^3+57\Theta^2+59\Theta+21) \\
& - 128t^9(2\Theta+3)(2\Theta+1)(\Theta+1)^2 \\
& \left\{ \begin{array}{cccccccc}
1/2 & 0 & -1/4 & -1/2 & -1 & (-1+\sqrt{-3})/4 & (-1-\sqrt{-3})/4 & \infty \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\
1 & 1/2 & 1 & 1/2 & 1 & 1 & 1 & 1 \\
1 & 1/2 & 1 & 1/2 & 1 & 3 & 3 & 1 \\
2 & 1 & 2 & 1 & 2 & 4 & 4 & 3/2
\end{array} \right\}
\end{aligned}$$

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